Homework 1

Problem 1. Consider the tossing a coin experiment. Suppose that $\Pr(H) = \Pr(T) = 1/2$. Define random variables X and Y as follows: $X(H) = 1$, $X(T) = 2$, $Y(H) = 2$, $Y(T) = 1$. Find CDFs of X and Y. Are X and Y equal in distribution (do they have the same CDFs.)? What is $\Pr(X = Y)$?

Problem 2. Suppose that the average distance between a random variable X and a constant c is measured by the function $\mathrm{E}(X - c)^2$.

1. Show that $\mathrm{E}(X - c)^2 = \mathrm{E}(X - \mathrm{E}(X))^2 + (\mathrm{E}(X) - c)^2$.
2. What value of c does minimize $\mathrm{E}(X - c)^2$?

Problem 3. Let X and Y be two continuously distributed random variables with the joint PDF given by

$$f_{X,Y}(x, y) = 1(0 < y < x < \sqrt{2}),$$

where $1(A)$ is the so-called *indicator function*:

$$1(A) = \begin{cases} 1, & \text{if condition } A \text{ is true,} \\ 0, & \text{if condition } A \text{ is false.} \end{cases}$$

Thus,

$$1(0 < y < x < \sqrt{2}) = \begin{cases} 1, & \text{if } 0 < y < x < \sqrt{2}, \\ 0, & \text{otherwise.} \end{cases}$$

1. Show that $f_{X,Y}$ is a PDF. Hint: Start by plotting the support of the distribution (the region where the PDF is non-zero).
2. Are X and Y statistically independent?
3. Find the marginal PDF of X.
4. Find the marginal PDF of Y.
5. Find the conditional PDF of Y given X.

Problem 4. Let $\boldsymbol{X}$, $\boldsymbol{Y}$, and $\boldsymbol{Z}$ be random vectors. Show that:

1. $\mathrm{Var}(\boldsymbol{X}) = \mathrm{E}(\boldsymbol{X}\boldsymbol{X}') - \mathrm{E}(\boldsymbol{X})\,\mathrm{E}(\boldsymbol{X}')$.
2. $\mathrm{Cov}(\boldsymbol{X}, \boldsymbol{Y}) = (\mathrm{Cov}(\boldsymbol{Y}, \boldsymbol{X}))'$.
3. $\mathrm{Var}(\boldsymbol{X} + \boldsymbol{Y}) = \mathrm{Var}(\boldsymbol{X}) + \mathrm{Var}(\boldsymbol{Y}) + \mathrm{Cov}(\boldsymbol{X}, \boldsymbol{Y}) + \mathrm{Cov}(\boldsymbol{Y}, \boldsymbol{X})$.
4. $\mathrm{Var}(\boldsymbol{\alpha} + \boldsymbol{\Gamma}\boldsymbol{X}) = \boldsymbol{\Gamma}\mathrm{Var}(\boldsymbol{X})\boldsymbol{\Gamma}'$, where $\boldsymbol{X}$ is a random n-vector, $\boldsymbol{\alpha}$ is a non-random k-vector, and $\boldsymbol{\Gamma}$ is a non-random $k \times n$ matrix.
5. $\mathrm{Cov}(\mathbf{A}\boldsymbol{X} + \mathbf{B}\boldsymbol{Y}, \mathbf{C}\boldsymbol{Z}) = \mathbf{A}(\mathrm{Cov}(\boldsymbol{X}, \boldsymbol{Z}))\mathbf{C}' + \mathbf{B}(\mathrm{Cov}(\boldsymbol{Y}, \boldsymbol{Z}))\mathbf{C}'$, where $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are non-random matrices.

Problem 5. Suppose that you had a new battery for your camera, and the life of the battery is a random variable X, with PDF

$$f_X(x) = k \times \exp\left(-\frac{x}{\beta}\right),$$

where $x > 0$ and β is a parameter. Assume now that t and s are non-negative real numbers.

(a). Use the properties of a PDF to determine the value of k.

(b). Find an expression for $\Pr(X \geq t)$.

(c). Find an expression for the conditional probability: $\Pr(X \geq t + s \mid X \geq s)$. Hint: Use $\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$.

(d). Suppose that your battery has already lasted for s weeks without dying. Based on your above answers, are you more concerned that the battery is about to die than you were when you first put it in the camera?

Problem 6. If $\mathbf{A}$ is a symmetric positive definite $k \times k$ matrix, then $\mathbf{I} - \mathbf{A}$ is positive definite if and only if $\mathbf{A}^{-1} - \mathbf{I}$ is positive definite, where $\mathbf{I}$ is the $k \times k$ identity matrix. Prove this result by considering the quadratic form $\boldsymbol{x}'(\mathbf{I} - \mathbf{A})\boldsymbol{x}$ and expressing $\boldsymbol{x}$ as $\mathbf{R}^{-1}\boldsymbol{z}$, where $\mathbf{R}$ is a symmetric matrix such that $\mathbf{A} = \mathbf{R}^2$. Then, extend this result to show that if $\mathbf{A}$ and $\mathbf{B}$ are symmetric positive definite matrices of the same dimensions, then $\mathbf{A} - \mathbf{B}$ is positive definite if and only if $\mathbf{B}^{-1} - \mathbf{A}^{-1}$ is positive definite. **Hint:** $\mathbf{A}$ is symmetric and positive definite, and therefore it can be written as $\mathbf{A} = \mathbf{C}\boldsymbol{\Lambda}\mathbf{C}'$, where $\boldsymbol{\Lambda}$ is the diagonal matrix composed of the positive eigenvalues, $\mathbf{C}$ is the matrix of eigenvectors, and $\mathbf{C}'\mathbf{C} = \mathbf{I}$. Now, one can write $\mathbf{A} = \mathbf{R}\mathbf{R}$, where $\mathbf{R} = \mathbf{C}\boldsymbol{\Lambda}^{1/2}\mathbf{C}'$. Furthermore, $\mathbf{A}^{-1} = \mathbf{R}^{-1}\mathbf{R}^{-1}$.

Problem 7. Let $\mathbf{A}$ be a symmetric matrix.

1. Show that the determinant of $\mathbf{A}$ is equal to the product of its eigenvalues.
2. Show that the trace of $\mathbf{A}$ is equal to the sum of its eigenvalues.
3. Show that $\mathbf{A}$ is positive definite (positive semidefinite) if and only if all its eigenvalues are positive (non-negative).

Problem 8. Let $\mathbf{B}$ be a symmetric and idempotent $n \times n$ matrix: $\mathbf{B}' = \mathbf{B}$ and $\mathbf{B}\mathbf{B} = \mathbf{B}$. Let $\lambda_1, \dots \lambda_n$ be the eigenvalues of $\mathbf{B}$.

1. Show that the eigenvalues of $\mathbf{B}$ are zeros and/or ones.
2. Show that $\operatorname{rank}(\mathbf{B}) = \sum_{i=1}^{n} \lambda_i = \operatorname{tr}(\mathbf{B})$.
3. Show that $\mathbf{B}$ is positive semidefinite.
4. Show that $\mathbf{I}_n - \mathbf{B}$ is also symmetric and idempotent.

Problem 9. Let $Y \in \{1, 2, 3, 4, 5, 6\}$ be the face number showing when a die is rolled. Define X as

$$X = \begin{cases} Y & \text{if } Y \text{ is even,} \\ 0 & \text{if } Y \text{ is odd.} \end{cases}$$

Find the best linear predictor $\mathcal{P}(Y|X)$ and the conditional expectation $\mathbb{E}(Y|X)$. Calculate $\mathbb{E}\left[(Y - \mathcal{P}(Y|X))^2\right]$ and $\mathbb{E}\left[(Y - \mathbb{E}(Y|X))^2\right]$.

Problem 10. Suppose that

$$\begin{aligned} Y =& \boldsymbol{X}'\boldsymbol{\beta} + e \\ \mathrm{E}(e|\boldsymbol{X}) =& 0 \end{aligned}$$

$$\mathrm{E}\left(e^2|\boldsymbol{X}\right) = \sigma^2(\boldsymbol{X}).$$

Consider two approximations to the conditional variance $\sigma^2(\boldsymbol{X})$:

$$\boldsymbol{\gamma}_1 \text{ minimizes } \mathrm{E}\left(\sigma^2(\boldsymbol{X}) - \boldsymbol{X}'\boldsymbol{\gamma}\right)^2$$

and

$$\boldsymbol{\gamma}_2 \text{ minimizes } \mathrm{E}\left(e^2 - \boldsymbol{X}'\boldsymbol{\gamma}\right)^2.$$

Show: $\boldsymbol{\gamma}_1 = \boldsymbol{\gamma}_2$.