Homework 3

Problem 1. (a) Prove the "Squeeze Rule": If $0 \le X_n \le Y_n$ and $Y_n \to_p 0$, then $X_n \to_p 0$; (b) Prove: $X_n \to_p 0$ if and only if $|X_n| \to_p 0$.

Problem 2. Provide a counter example to show that $X_n \to_d X$ and $Y_n \to_d Y$ does not imply $X_n + Y_n \to_d X + Y$. Hint: Consider an iid random sample $X_1, ..., X_n$ with $\mathbb{E}X_1 = 0$ and $n^{1/2}\overline{X}_n$ and $-n^{1/2}\overline{X}_n$.

Problem 3. Let $\widehat{\boldsymbol{\theta}}_n = \left(\widehat{\theta}_{n,1}, \ldots, \widehat{\theta}_{n,k}\right)'$ be an estimator of the k-vector of parameters $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_k)'$. Suppose that $\widehat{\boldsymbol{\theta}}_n \to_p \boldsymbol{\theta}$, and $n^{1/2}\left(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}\right) \to_d W \sim N(0, \boldsymbol{\Sigma})$, where $\boldsymbol{\Sigma}$ is a positive definite $k \times k$ matrix. Use the delta method or CMT to find the (non-degenerate, i.e., not a constant) asymptotic distributions of the following quantities after a suitable normalization. "Suitable normalization" means subtraction of a constant and/or multiplication by a constant (could be dependent on n).

1. $n^{1/2}\left(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}\right)'\boldsymbol{c}$ where $\boldsymbol{c} \in \mathbb{R}^k$ is a vector of constants.
2. $\widehat{\theta}_{n,1}$.
3. $n\left(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}\right)'\left(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}\right)$.
4. $\widehat{\theta}_{n,1} - \widehat{\theta}_{n,2}$.
5. $\widehat{\theta}_{n,1}\widehat{\theta}_{n,2}/\widehat{\theta}_{n,3}$, provided that $\theta_3 \neq 0$.

Problem 4. Suppose that $\hat{\theta}_n \to_p \theta$ and $\hat{\beta}_n \to \beta$, where θ and β are two scalar parameters. Without relying on Slutsky's Theorem, show:

1. $c\hat{\theta}_n \to_p c\theta$, where c is a constant.
2. $\hat{\theta}_n\hat{\beta}_n \to_p \theta\beta$.

Problem 5. Suppose that $\mathbb{E}\left(\hat{\theta}_n\right) \to \theta$ and $\mathrm{Var}(\hat{\theta}_n) \to 0$ as $n \to \infty$. Show that $\hat{\theta}_n \to_p \theta$.

Problem 6. Consider the linear model (with independently and identically distributed (i.i.d.) observations):

$$Y_i = \beta_0 + \beta_1 X_{1,i} + \beta_2 X_{2,i} + U_i$$

with $\mathbb{E}U_i = \mathbb{E}U_i X_{1,i} = \mathbb{E}U_i X_{2,i} = 0$. Suppose we know that $\beta_2 = \beta_1$ and conduct a constrained LS estimation of β_1:

$$\min_{b_0, b_1} \sum_{i=1}^{n} \left(Y_i - b_0 - b_1 X_{1,i} - b_1 X_{2,i}\right)^2.$$

1. Find the expression for the constrained LS estimator $\left(\widehat{\beta}_0, \widehat{\beta}_1\right)$ that solve the above minimization problem.
2. Assume that the restriction $\beta_2 = \beta_1$ is true. Derive the large-sample (asymptotic) distribution of $\widehat{\beta}_1$.

Problem 7. Suppose we observe the i.i.d. random sample $\{(Y_i, X_i)\}_{i=1}^n$ with X_i being a scalar. Take the linear model

$$\begin{aligned} Y_i &= X_i\beta + e_i \\ \mathbb{E}\left(e_i|X_i\right) &= 0. \end{aligned}$$

Consider the estimator

$$\widehat{\beta} = \frac{\sum_{i=1}^n X_i^3 Y_i}{\sum_{i=1}^n X_i^4}.$$

Find the asymptotic distribution of $\sqrt{n}\left(\widehat{\beta} - \beta\right)$.

Problem 8. Let $\{\theta_n : n \geq 1\}$ be a random sequence such that $\Pr\left(\theta_n = 0\right) = (n-1)/n$, and $\Pr\left(\theta_n = n^2\right) = 1/n$. Note that the only possible values for θ_n are zero and n^2.

1. Show that $\lim_{n\to\infty} \mathbb{E}\theta_n = \infty$.
2. Does θ_n converge in probability to some limit? If yes, prove. If not, explain why.