Statistical Learning

Homework 4

Part 1: Conceptual Questions

Problem 1. ISL (second edition), Question 1 on Page 321.

Solution. (a) For $x \le \xi$, we have

$$f(x) = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3.$$

Take $a_1 = \beta_0$, $b_1 = \beta_1$, $c_1 = \beta_2$ and $d_1 = \beta_3$.
(b) For $x > \xi$, we have

$$\begin{aligned} f(x) &= \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \beta_4 (x-\xi)^3 \\ &= \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \beta_4 \left(x^3 - \xi^3 + 3x\xi^2 - 3x^2\xi\right) \\ &= \left(\beta_0 - \beta_4\xi^3\right) + \left(\beta_1 + 3\beta_4\xi^2\right) x + (\beta_2 - 3\beta_4\xi) x^2 + (\beta_3 + \beta_4) x^3. \end{aligned}$$

Take $a_2 = \beta_0 - \beta_4\xi^3$, $b_2 = \beta_1 + 3\beta_4\xi^2$, $c_2 = \beta_2 - 3\beta_4\xi$ and $d_2 = \beta_3 + \beta_4$.
(c)

$$\begin{aligned} \lim_{x\uparrow\xi} f(x) = \lim_{x\uparrow\xi} f_1(x) = f_1(\xi) = \beta_0 + \beta_1\xi + \beta_2\xi^2 + \beta_3\xi^3 \\ \lim_{x\downarrow\xi} f(x) = \lim_{x\downarrow\xi} f_2(x) = f_2(\xi) = \beta_0 + \beta_1\xi + \beta_2\xi^2 + \beta_3\xi^3, \end{aligned}$$

where "$x \uparrow \xi$" means that x approaches ξ from left.
(d) It is easy to check that

$$\begin{aligned} f_1'(x) = \beta_1 + 2\beta_2 x + 3\beta_3 x^2 \\ f_2'(x) = \left(\beta_1 + 3\beta_4\xi^2\right) + 2(\beta_2 - 3\beta_4\xi) x + 3(\beta_3 + \beta_4) x^2. \end{aligned}$$

Then,

$$\begin{aligned} \lim_{\Delta\uparrow 0} \frac{f(\xi+\Delta) - f(\xi)}{\Delta} = \lim_{x\uparrow\xi} f'(x) = \lim_{x\uparrow\xi} f_1'(x) = f_1'(\xi) = \beta_1 + 2\beta_2\xi + 3\beta_3\xi^2 \\ \lim_{\Delta\downarrow 0} \frac{f(\xi+\Delta) - f(\xi)}{\Delta} = \lim_{x\downarrow\xi} f'(x) = \lim_{x\downarrow\xi} f_2'(x) = f_2'(\xi) = \beta_1 + 2\beta_2\xi + 3\beta_3\xi^2, \end{aligned}$$

where the first equality follows from the mean value theorem: for $\Delta > 0$, $(f(\xi+\Delta) - f(\xi))/\Delta = f'(\dot{x})$, where $\xi < \dot{x} < \xi + \Delta$ and $\dot{x} \downarrow \xi$ as $\Delta \downarrow 0$.
(e) It is easy to check that

$$\begin{aligned} f_1''(x) = 2\beta_2 + 6\beta_3 x \\ f_2''(x) = 2(\beta_2 - 3\beta_4\xi) + 6(\beta_3 + \beta_4) x. \end{aligned}$$

Then,

$$\lim_{\Delta\uparrow 0}\frac{f'(\xi+\Delta)-f'(\xi)}{\Delta}=\lim_{x\uparrow\xi}f''(x)=\lim_{x\uparrow\xi}f_1''(x)=f_1''(\xi)=2\beta_2+6\beta_3\xi$$
$$\lim_{\Delta\downarrow 0}\frac{f'(\xi+\Delta)-f'(\xi)}{\Delta}=\lim_{x\downarrow\xi}f''(x)=\lim_{x\downarrow\xi}f_2''(x)=f_2''(\xi)=2\beta_2+6\beta_3\xi.$$

Problem 2. ISL (second edition), Question 2 on Page 322.
(a) Infinitely large penalty is imposed and forces $g=0$ (a constant function zero). In this case, $\hat{g}=0$.
(b) Penalty forces $g^{(1)}=0$ and therefore $\hat{g}=m^*$, where

$$m^*=\underset{m}{\operatorname{argmin}}\sum_{i=1}^{n}(y_i-m)^2.$$

(c) Penalty forces $g^{(2)}=0$ and therefore, $\hat{g}(x)=a^*+b^*x$, where

$$(a^*,b^*)=\underset{a,b}{\operatorname{argmin}}\sum_{i=1}^{n}(y_i-(a+bx_i))^2.$$

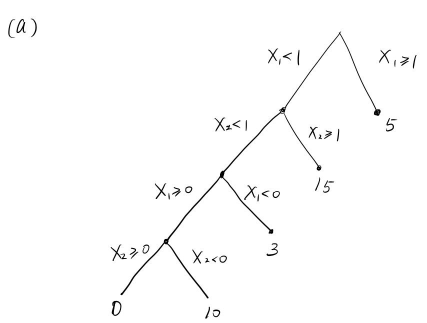
(d) Penalty forces $g^{(3)}=0$ and therefore, $\hat{g}(x)=a^*+b^*x+c^*x^2$, where

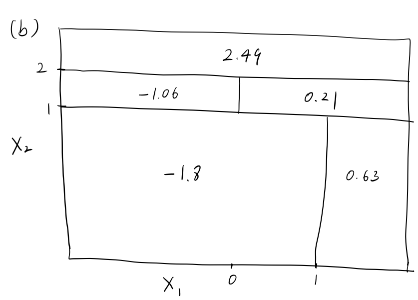
$$(a^*,b^*,c^*)=\underset{a,b,c}{\operatorname{argmin}}\sum_{i=1}^{n}\left(y_i-\left(a+bx_i+cx_i^2\right)\right)^2.$$

(e) Zero penalty and therefore, $\hat{g}$ interpolates the data, i.e., $\hat{g}(x_i)=y_i$, for all i.

Problem 3. ISL (second edition), Question 4 on Page 362.

Solution.





Part 2: Applied Questions

Problem 4. ISL (second edition), Question 9 on Page 324.

Problem 5. ISL (second edition), Question 8 on Page 363.