Econometrics

Homework 1

Problem 1. Prove the following results:

1. If c is a constant, then $Cov\left(X, c\right) = 0$.
2. $Cov\left(X, X\right) = Var\left(X\right)$.
3. $Cov\left(X, Y\right) = Cov\left(Y, X\right)$.
4. $Cov\left(a_1 + b_1 X, a_2 + b_2 Y\right) = b_1 b_2 Cov\left(X, Y\right)$, where a_1, a_2, b_1, and b_2 are some constants.
5. $Var\left(X + Y\right) = Var\left(X\right) + Var\left(Y\right) + 2Cov\left(X, Y\right)$.
6. $Var\left(X - Y\right) = Var\left(X\right) + Var\left(Y\right) - 2Cov\left(X, Y\right)$.

Problem 2. Let X and Y be two discrete random variables. The set of possible values for X is $\{x_1, \ldots, x_n\}$, and the set of possible values for Y is $\{y_1, \ldots, y_m\}$. The joint PMF is given by

$$p_{ij}^{X,Y} = P\left(X = x_i, Y = y_j\right), \qquad i = 1, \ldots n; j = 1, \ldots, m.$$

Show that if X and Y are independent then $Cov\left(X, Y\right) = 0$.

Problem 3. Let $\{x_i : i = 1, \ldots, n\}$ and $\{y_i : i = 1, \ldots, n\}$ be two sequences. Define the averages

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i,$$
$$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i.$$

1. Show that $\sum_{i=1}^{n}\left(x_i - \bar{x}\right) = 0$.
2. Using the result in part (a), show that

$$\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2 = \sum_{i=1}^{n} x_i\left(x_i - \bar{x}\right), \text{ and}$$
$$\sum_{i=1}^{n}\left(x_i - \bar{x}\right)\left(y_i - \bar{y}\right) = \sum_{i=1}^{n} y_i\left(x_i - \bar{x}\right) = \sum_{i=1}^{n} x_i\left(y_i - \bar{y}\right).$$

Problem 4. The distribution of income X within countries is often modeled as a Pareto distribution with parameter $A > 0$ and $\gamma > 0$, with CDF

$$F_X(x) = \begin{cases} 0 & \text{if } x \leq A \text{ and} \\ 1 - \left(\frac{x}{A}\right)^{-\gamma}, & \text{if } x > A. \end{cases}$$

Assume $\gamma > 1$ and calculate $E[X]$.

Problem 5. Show

$$E[(Y - E[Y \mid X])(E[Y \mid X] - E[Y])] = 0.$$

Problem 6. Show that if $E[Y \mid X] = E[Y]$, then $Cov(X, Y) = 0$.

Problem 7. The conditional PDF of Y given X is

$$f_{Y|X}(y \mid x) = \frac{y + x}{1/2 + x},$$

for $0 < y < 1$. Find $\mathrm{E}[Y \mid X = x]$.

Problem 8. A Monet expert is given a painting purported to be a lost Monet. He is asked to assess the chances that it is genuine and has the following information:
• In general, only 1% of the "found" paintings he receives turn out to be genuine, an event well call G
• "Found" paintings have a different frequency of use of certain pigments than genuine Monets do: (i) cadmium yellow Y appears in 20% "found" paintings, but only 10% genuine ones (ii) raw umber U appears in 80% of "found" paintings, but only 40% of genuine ones (iii) burnt sienna S appears in 40% of "found" paintings, but 60% of genuine paintings
• This particular painting uses burnt sienna, but not cadmium yellow or raw umber.
What is the probability that this particular painting is genuine? Do we have to make any additional assumptions to answer the question?

Problem 9. For any given two random variables X and Y, we define

$$\mathrm{Var}[Y \mid X] = \mathrm{E}\left[(Y - \mathrm{E}[Y \mid X])^2 \mid X\right].$$

Suppose that $\mathrm{E}[Y \mid X] = 1/4$ and $\mathrm{E}[Y^2 \mid X] = 1/8$. Then show that for any function g, $\mathrm{Var}[Y \mid g(X)] = 1/16$. Use the following facts: for any function g, $\mathrm{E}[\mathrm{E}[Y \mid g(X)] \mid X] = \mathrm{E}[Y \mid g(X)]$ and $\mathrm{E}[\mathrm{E}[Y \mid X] \mid g(X)] = \mathrm{E}[Y \mid g(X)]$.