Econometrics

Homework 2

Problem 1. The LS objective function discussed in class is

$$Q(a,b) = \sum_{i=1}^{n} (Y_i - a - bX_i)^2 .$$

Now consider a modification of it:

$$\widetilde{Q}(a,b) = \left[\sum_{i=1}^{n} (Y_i - a - bX_i)\right]^2 .$$

Let $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ be the minimizer of $\widetilde{Q}$. Show that $\widetilde{Q}\left(\widetilde{\alpha}, \widetilde{\beta}\right) = 0$. Hint: You do not need to derive the first order conditions.

Problem 2. Suppose that you had a new battery for your camera, and the life of the battery is a random variable X, with PDF

$$f_X(x) = k \times \exp\left(-\frac{x}{\beta}\right),$$

where $x > 0$ and β is a parameter. Assume now that t and s are non-negative real numbers.
(a). Use the properties of a PDF to determine the value of k.
(b). Find an expression for $\Pr[X \geq t]$.
(c). Find an expression for the conditional probability: $\Pr[X \geq t + s \mid X \geq s]$. Hint: Use $\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$.
(d). Suppose that your battery has already lasted for s weeks without dying. Based on your above answers, are you more concerned that the battery is about to die than you were when you first put it in the camera?

Problem 3. Suppose that X is a continuous random variable with a strictly increasing and differentiable CDF F_X and PDF $f_X = F'_X$. (a) Show that $\mathrm{E}\left[(X-a)^2\right]$ is minimized at $a = \mathrm{E}[X]$. (b) Show that $\mathrm{E}[|X-a|]$ is minimized at $a = F_X^{-1}(1/2)$ (the median). Hint: $\mathrm{E}[|X-a|] = \int_{-\infty}^{a} (a-x) f_X(x)\, dx + \int_{a}^{\infty} (x-a) f_X(x)\, dx$.

Problem 4. Consider a simple regression model with no intercept:

$$Y_i = \beta X_i + U_i,$$

and assume that for all $i = 1, \ldots, n$:

$$\begin{aligned}
\mathrm{E}[U_i \mid X_1, \ldots, X_n] &= 0, \\
\mathrm{E}\left[U_i^2 \mid X_1, \ldots, X_n\right] &= \sigma^2, \\
\mathrm{E}[U_i U_j \mid X_1, \ldots, X_n] &= 0 \text{ for } i \neq j.
\end{aligned}$$

1. Show that the OLS estimator of β is
$$\hat{\beta} = \frac{\sum_{i=1}^{n} X_i Y_i}{\sum_{i=1}^{n} X_i^2}.$$
2. Show that for the fitted residuals $\hat{U}_i = Y_i - \hat{\beta} X_i$,
$$\sum_{i=1}^{n} \hat{U}_i X_i = 0.$$
3. Is it necessarily true that $\sum_{i=1}^{n} \hat{U}_i = 0$? Explain.
4. Show that $\hat{\beta}$ in part (a) is unbiased.
5. Show that conditionally on X's:
$$\text{Var}\left[\hat{\beta}\right] = \frac{\sigma^2}{\sum_{i=1}^{n} X_i^2}.$$

Problem 5. The following table gives the joint probability distribution between employment status and college graduation among those either employed or unemployed.

	$Y = 1$ (employed)	$Y = 0$ (unemployed)
$X = 0$ (no college)	0.05	0.6
$X = 1$ (college graduate)	0.33	0.02

1. What is the mean of Y?
2. What is the mean of X?
3. What is the conditional mean of Y given $X = 0$?
4. What is the covariance of X and Y?
5. Are X and Y independent?
6. What is the probability of being employed?
7. What is the variance of X?
8. Suppose that you had a sample (X_i, Y_i), $i = 1, ..., n$ drawn from the joint distribution in the table. And now suppose that you estimate the following model by using this sample:
$$Y_i = \alpha + \beta X_i + U_i, \ \text{E}\left[U_i \mid X_i\right] = 0.$$
What is the true value of the parameter β?

Problem 6. Suppose you have a sample (X_i, Y_i), $i = 1, ..., n$ and estimate the linear model
$$Y_i = \beta_0 + \beta_1 X_i + U_i, \ \text{E}\left[U_i \mid X_i\right] = 0$$
by OLS. The OLS estimator for the slope is $\widehat{\beta_1}$. Now estimate another linear model
$$X_i = \gamma_0 + \gamma_1 Y_i + V_i, \ \text{E}\left[V_i \mid Y_i\right] = 0.$$
The OLS estimator for the slope is $\widehat{\gamma_1}$ Is it true that$\widehat{\gamma_1} = \widehat{\beta_1}^{-1}$?

Problem 7. (Wooldridge Problem 2.8) Consider the standard simple regression model $Y_i = \beta_0 + \beta_1 X_i + U_i$ under the following assumptions: (1) (X_i, Y_i), $i = 1, ..., n$ are i.i.d. observations (2) $\mathrm{E}[U_i \mid X_i] = 0$ and $\mathrm{Var}[U_i \mid X_i] = \sigma^2$ (conditional homoskedasticity). The usual OLS estimators $\hat{\beta}_0$ and $\hat{\beta}_1$ are unbiased for β_0 and β_1. Let $\tilde{\beta}_1$ be the estimator of β_1 obtained by assuming the intercept is zero: $\tilde{\beta}_1$ is the minimizer of

$$\min_b \sum_{i=1}^n (Y_i - bX_i)^2 .$$

Hint: The estimator $\tilde{\beta}_1$ is constructed under the assumption that $\beta_0 = 0$. When answering the question, keep in mind that this assumption can be false and the true value of β_0 can be different from zero.

1. Find $\mathrm{E}\left[\tilde{\beta}_1\right]$ (the conditional expectation) in terms of the X's, β_0, and β_1. Verify that $\tilde{\beta}_1$ is unbiased for β_1 when the population intercept (β_0) is zero. Are there other cases where $\tilde{\beta}_1$ is unbiased?

2. Find the variance of $\tilde{\beta}_1$. (Hint: The variance does not depend on β_0.)

3. Show that $\mathrm{Var}\left[\tilde{\beta}_1\right] \leq \mathrm{Var}\left[\hat{\beta}_1\right]$ by showing the following fact: for any sample of data, $\sum_{i=1}^n X_i^2 \geq \sum_{i=1}^n \left(X_i - \bar{X}\right)^2$, with strict inequality unless $\bar{X} = 0$.