Appendix to "Inference on the Distribution of Individual Treatment Effects in Nonseparable Triangular Models"

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The appendix contains two sections. Appendix A is devoted to the proofs of the results in Section 3. Appendix B is devoted to the proofs of the results in Section 4.

Notation. Denote $U_i \coloneqq (\epsilon_i, D_i, Z_i)$ and $U \coloneqq (\epsilon, D, Z)$. For notational simplicity, we suppress the dependence of n_x on x and write n instead. Since the domains are often clear from the definitions, we suppress the dependence of the sup-norm on the domain and write $\|\cdot\|_\infty$ for simplicity. Similarly, we suppress the dependence of $BL_1(\mathbb{D})$ on the domain $\mathbb{D}$ and write BL_1 for simplicity. For a subset A of $\mathbb{D}$ endowed with a norm $\|\cdot\|$ and $\delta > 0$, let $A^\delta \coloneqq \{x \in \mathbb{D} : \exists y \in A \text{ such that } \|x - y\| < \delta\}$ denote the δ-enlargement of A. It is easy to see that A^δ must be an open set. "$\lesssim$" denotes inequality up to a universal constant. "With probability approaching one" is abbreviated to "wpa1". $f|_B$ denotes the restriction of a function f with a domain A on a sub-domain $B \subseteq A$.

A Proofs of results in Section 3

We denote $\mathbb{P}_n f \coloneqq n^{-1} \sum_{i=1}^n f(U_i)$, $\mathbb{P} f \coloneqq \mathrm{E}[f(U) \mid X = x]$ and $\mathbb{G}_n \coloneqq \sqrt{n}(\mathbb{P}_n - \mathbb{P})$. Let

$$\gamma_{dx}(U_i \mid v, h, h') \coloneqq \left\{ \mathbb{1}\left(\mathit{\Delta}_x(\epsilon_i) + \frac{h(\epsilon_i)}{\sqrt{n} \cdot \varsigma_{dx}(\epsilon_i)} \leq v \right) - \mathbb{1}\left(\mathit{\Delta}_x(\epsilon_i) + \frac{h'(\epsilon_i)}{\sqrt{n} \cdot \varsigma_{dx}(\epsilon_i)} \leq v \right) \right\} \mathbb{1}(D_i = d')$$

and $\gamma_x(U_i \mid v, h, h') \coloneqq \sum_{d \in \{0,1\}} \gamma_{dx}(U_i \mid v, h, h')$. Then we can write

$$\breve{F}_{\Delta|X}(v \mid x) - \widetilde{F}_{\Delta|X}(v \mid x) = \mathbb{P}_n \gamma_x\left(\cdot \mid v, \sqrt{n} \cdot H_x, 0\right)$$

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and

$$\breve{F}_{\Delta|X}(v \mid x) - \widetilde{F}_{\Delta|X}(v \mid x) - \sum_{d \in \{0,1\}} \left\{ \Psi_{dx}\left(\Delta_x + \frac{H_x}{\varsigma_{dx}}\right) - \Psi_{dx}\Delta_x \right\}(v) = \frac{1}{\sqrt{n}} \cdot \mathbb{G}_n \gamma_x\left(\cdot \mid v, \sqrt{n} \cdot H_x, 0\right). \tag{A1}$$

Using these notations, we show the following lemma.

Lemma 1. *Under the assumptions in the statement of Theorem 1, (16) holds.*

Proof of Lemma 1. The conclusion of the lemma follows from (A1) and $\mathbb{G}_n \gamma_{dx}\left(\cdot \mid v, \sqrt{n} \cdot H_x, 0\right) = o_p(1)$, uniformly in $[\underline{v}_x, \overline{v}_x]$, for $d \in \{0, 1\}$. The latter result is equivalent to the claim that for all $\varepsilon > 0$,

$$\Pr\left[\sup_{v \in [\underline{v}_x, \overline{v}_x]} \left|\mathbb{G}_n \gamma_{dx}\left(\cdot \mid v, \sqrt{n} \cdot H_x, 0\right)\right| > \varepsilon\right] \leq \varepsilon, \tag{A2}$$

when n is sufficiently large. Note that $\mathbb{H}_x$ concentrates on the space $C[\underline{\epsilon}_x, \overline{\epsilon}_x]$, which is a closed subset of $\ell^\infty[\underline{\epsilon}_x, \overline{\epsilon}_x]$. Fix some $\varepsilon > 0$. It follows from the tightness of $\mathbb{H}_x$ that there exists some compact set $K \subseteq C[\underline{\epsilon}_x, \overline{\epsilon}_x]$ such that $\Pr[\mathbb{H}_x \in K] > 1 - \varepsilon/2$. By this result, (13) and the portmanteau lemma that for all $\delta > 0$ when n is sufficiently large,

$$\Pr\left[\sqrt{n} \cdot H_x \in K^{\delta/2}\right] \geq \Pr\left[\mathbb{H}_x \in K^{\delta/2}\right] > 1 - \frac{\varepsilon}{2}. \tag{A3}$$

Let $B(h_0, \delta) := \{h \in \ell^\infty[\underline{\epsilon}_x, \overline{\epsilon}_x] : \|h - h_0\|_\infty < \delta\}$ denote the open ball of radius δ in $\ell^\infty[\underline{\epsilon}_x, \overline{\epsilon}_x]$ centered around h_0. Note that K is also compact in the larger space $\ell^\infty[\underline{\epsilon}_x, \overline{\epsilon}_x]$. Then by the compactness of K, there exists some $p_\delta \in \mathbb{N}$ and $h_1, ..., h_{p_\delta} \in K$ such that $K \subseteq \bigcup_{k=1}^{p_\delta} B(h_k, \delta/2)$. By the triangle inequality, $K^{\delta/2} \subseteq \bigcup_{k=1}^{p_\delta} B(h_k, \delta)$. Therefore, by this result and (A3), for each $\delta > 0$,

$$\Pr\left[\sqrt{n} \cdot H_x \in \bigcup_{k=1}^{p_\delta} B(h_k, \delta)\right] > 1 - \frac{\varepsilon}{2}, \tag{A4}$$

when n is sufficiently large. By the triangle inequality,

$$\begin{aligned} & \sup_{h \in \bigcup_{k=1}^{p_\delta} B(h_k,\delta)} \sup_{v \in [\underline{v}_x, \overline{v}_x]} \left|\mathbb{G}_n \gamma_{dx}(\cdot \mid v, h, 0)\right| \\ \leq & \sup_{v \in [\underline{v}_x, \overline{v}_x]} \max_{k=1,\ldots,p_\delta} \sup_{h \in B(h_k,\delta)} \left|\mathbb{G}_n \gamma_{dx}(\cdot \mid v, h, 0)\right| \\ \leq & \sup_{v \in [\underline{v}_x, \overline{v}_x]} \max_{k=1,\ldots,p_\delta} \sup_{h \in B(h_k,\delta)} \left|\mathbb{G}_n \gamma_{dx}(\cdot \mid v, h, h_k)\right| + \sup_{v \in [\underline{v}_x, \overline{v}_x]} \max_{k=1,\ldots,p_\delta} \left|\mathbb{G}_n \gamma_{dx}(\cdot \mid v, h_k, 0)\right|. \end{aligned} \tag{A5}$$

Let $\underline{\varsigma}_{dx} := \inf_{e \in [\underline{\epsilon}_x, \overline{\epsilon}_x]} |\varsigma_{dx}(e)|$. We have $\underline{\varsigma}_{dx} > 0$ under Assumption 1(g). By using the elementary equality

$$\begin{aligned} \mathbb{1}(u \leq x) - \mathbb{1}(u \leq 0) &= \mathbb{1}(x > 0)\mathbb{1}(0 < u \leq x) - \mathbb{1}(x < 0)\mathbb{1}(x < u \leq 0) \\ &= \mathbb{1}(x > 0)\mathbb{1}(0 < u \leq |x|) - \mathbb{1}(x < 0)\mathbb{1}(-|x| < u \leq 0), \end{aligned} \tag{A6}$$

and the triangle inequality, for all $h \in B(h_k, \delta)$, we have

$$\begin{aligned}
& \left|\gamma_{dx}(U_i \mid v, h, h_k)\right| \\
\leq\ & \left|\mathbb{1}\left(\Delta_x(\epsilon_i) + \frac{h(\epsilon_i)}{\sqrt{n} \cdot \varsigma_{dx}(\epsilon_i)} \leq v\right) - \mathbb{1}\left(\Delta_x(\epsilon_i) + \frac{h_k(\epsilon_i)}{\sqrt{n} \cdot \varsigma_{dx}(\epsilon_i)} \leq v\right)\right| \mathbb{1}(D_i = d') \\
\leq\ & \mathbb{1}\left(v - \frac{\delta}{\sqrt{n} \cdot \underline{\varsigma}_{dx}} < \Delta_x(\epsilon_i) + \frac{h_k(\epsilon_i)}{\sqrt{n} \cdot \varsigma_{dx}(\epsilon_i)} \leq v + \frac{\delta}{\sqrt{n} \cdot \underline{\varsigma}_{dx}}\right) \mathbb{1}(D_i = d') \\
=:\ & \Gamma_{dx}(U_i \mid v, \delta, h_k). \qquad \text{(A7)}
\end{aligned}$$

Therefore, by this result and the triangle inequality,

$$\sup_{h \in B(h_k, \delta)} \left|\mathbb{G}_n \gamma_{dx}(\cdot \mid v, h, h_k)\right| \leq \left|\mathbb{G}_n \Gamma_{dx}(\cdot \mid v, \delta, h_k)\right| + 2\sqrt{n} \cdot \mathbb{P}\Gamma_{dx}(\cdot \mid v, \delta, h_k).$$

Now by this result and (A5),

$$\sup_{h \in \bigcup_{k=1}^{p_\delta} B(h_k, \delta)} \sup_{v \in [\underline{v}_x, \overline{v}_x]} \left|\mathbb{G}_n \gamma_{dx}(\cdot \mid v, h, 0)\right| \leq T_n(\delta) + \sup_{(v,h) \in [\underline{v}_x, \overline{v}_x] \times K} 2\sqrt{n} \cdot \mathbb{P}\Gamma_{dx}(\cdot \mid v, \delta, h), \qquad \text{(A8)}$$

where

$$T_n(\delta) := \sup_{v \in [\underline{v}_x, \overline{v}_x]} \max_{k=1,\ldots,p_\delta} \left|\mathbb{G}_n \gamma_{dx}(\cdot \mid v, h_k, 0)\right| + \sup_{v \in [\underline{v}_x, \overline{v}_x]} \max_{k=1,\ldots,p_\delta} \left|\mathbb{G}_n \Gamma_{dx}(\cdot \mid v, \delta, h_k)\right|.$$

By Kosorok (2007, Lemma 9.7, iii and vi), for any $h \in K$,

$$\left\{e \mapsto \mathbb{1}\left(\Delta_x(e) + \frac{h(e)}{\sqrt{n} \cdot \varsigma_{dx}(e)} \leq v\right) - \mathbb{1}\left(\Delta_x(e) \leq v\right) : v \in [\underline{v}_x, \overline{v}_x]\right\}$$

is a Vapnik-Červonenkis(VC) class of functions (see, e.g., Kosorok, 2007, Chapter 9.1.1 for its definition) which has a VC index independent of n and is uniformly bounded by the constant envelope 2. Note that since K is a compact subset of $C[\underline{\epsilon}_x, \overline{\epsilon}_x]$, we have $\sup_{h \in K} \|h\|_\infty < M$ for some $M > 0$ (Dudley, 2002, Theorem 2.4.7). By Kosorok (2007, Theorem 9.3) and Chen and Kato (2020, Corollary 5.5), we have

$$\mathrm{E}\left[\sup_{v \in [\underline{v}_x, \overline{v}_x]} \left|\mathbb{G}_n \gamma_{dx}(\cdot \mid v, h, 0)\right|\right] \lesssim \sigma_\gamma \sqrt{\log(n)} + \frac{\log(n)}{\sqrt{n}}, \qquad \text{(A9)}$$

where

$$\begin{aligned}
\sigma_\gamma^2 &:= \sup_{v \in [\underline{v}_x, \overline{v}_x]} \mathbb{P}\gamma_{dx}^2(\cdot \mid v, h, 0) \\
&\leq \sup_{v \in [\underline{v}_x, \overline{v}_x]} \left\{F_{\Delta|X}\left(v + \frac{M}{\sqrt{n} \cdot \underline{\varsigma}_{dx}}\right) - F_{\Delta|X}\left(v - \frac{M}{\sqrt{n} \cdot \underline{\varsigma}_{dx}}\right)\right\} \\
&= O\left(n^{-1/2}\right),
\end{aligned}$$

where the inequality follows from (A6) and the triangle inequality. Therefore, for all $h \in K$, $\sup_{v \in [\underline{v}_x, \overline{v}_x]} \left| \mathbb{G}_n \gamma_{dx} \left(\cdot \mid v, h, 0 \right) \right| = o_p\left(1\right)$ and this result implies that for all $\delta > 0$,

$$\sup_{v \in [\underline{v}_x, \overline{v}_x]} \max_{k=1,\ldots,p_\delta} \left| \mathbb{G}_n \gamma_{dx} \left(\cdot \mid v, h_k, 0 \right) \right| = o_p\left(1\right). \tag{A10}$$

By similar arguments, for all $h \in K$ and $\delta > 0$, $\sup_{v \in [\underline{v}_x, \overline{v}_x]} \left| \mathbb{G}_n \varGamma_{dx} \left(\cdot \mid v, \delta, h \right) \right| = o_p\left(1\right)$ and this result implies that for all $\delta > 0$,

$$\sup_{v \in [\underline{v}_x, \overline{v}_x]} \max_{k=1,\ldots,p_\delta} \left| \mathbb{G}_n \varGamma_{dx} \left(\cdot \mid v, \delta, h_k \right) \right| = o_p\left(1\right). \tag{A11}$$

(A11) and (A10) imply that for each $\delta > 0$, $T_n\left(\delta\right) = o_p\left(1\right)$.

Now we show that

$$\lim_{\delta \downarrow 0} \limsup_{n \uparrow \infty} \sup_{(v,h) \in [\underline{v}_x, \overline{v}_x] \times K} \sqrt{n} \cdot \mathbb{P} \varGamma_{dx} \left(\cdot \mid v, \delta, h \right) = 0. \tag{A12}$$

By change of variables $u = \sqrt{n}\left(\varDelta_{x,j}\left(e\right) - v\right)$,

$$\begin{aligned}
& \sqrt{n} \cdot \mathbb{P} \varGamma_{dx} \left(\cdot \mid v, \delta, h \right) \\
= \; & \sqrt{n} \sum_{j=1}^{m} \int_{\epsilon_{x,j-1}}^{\epsilon_{x,j}} \mathbb{1}\left(v - \frac{\delta}{\sqrt{n} \cdot \underline{\varsigma}_{dx}} \leq \varDelta_{x,j}\left(e\right) + \frac{h\left(e\right)}{\sqrt{n} \cdot \varsigma_{dx}\left(e\right)} \leq v + \frac{\delta}{\sqrt{n} \cdot \underline{\varsigma}_{dx}} \right) f_{(\epsilon,D)|X}\left(e, d' \mid x\right) \mathrm{d}e \\
= \; & \sum_{j=1}^{m} \int_{\sqrt{n}\left(\varDelta_{x,j}\left(\epsilon_{x,j-1}\right) - v\right)}^{\sqrt{n}\left(\varDelta_{x,j}\left(\epsilon_{x,j}\right) - v\right)} \mathbb{1}\left(u \geq -\frac{h\left(\varDelta_{x,j}^{-1}\left(n^{-1/2}u + v\right)\right)}{\varsigma_{dx}\left(\varDelta_{x,j}^{-1}\left(n^{-1/2}u + v\right)\right)} - \frac{\delta}{\underline{\varsigma}_{dx}} \right) \\
& \times \mathbb{1}\left(u \leq -\frac{h\left(\varDelta_{x,j}^{-1}\left(n^{-1/2}u + v\right)\right)}{\varsigma_{dx}\left(\varDelta_{x,j}^{-1}\left(n^{-1/2}u + v\right)\right)} + \frac{\delta}{\underline{\varsigma}_{dx}} \right) \rho_{d'x,j}\left(n^{-1/2}u + v\right) \mathrm{d}u.
\end{aligned} \tag{A13}$$

Assume that $\varDelta_{x,j}$ is increasing without loss of generality. We have

$$\begin{aligned}
& \int_{\sqrt{n}\left(\varDelta_{x,j}\left(\epsilon_{x,j-1}\right) - v\right)}^{\sqrt{n}\left(\varDelta_{x,j}\left(\epsilon_{x,j}\right) - v\right)} \mathbb{1}\left(u \geq -\frac{h\left(\varDelta_{x,j}^{-1}\left(n^{-1/2}u + v\right)\right)}{\varsigma_{dx}\left(\varDelta_{x,j}^{-1}\left(n^{-1/2}u + v\right)\right)} - \frac{\delta}{\underline{\varsigma}_{dx}} \right) \\
& \times \mathbb{1}\left(u \leq -\frac{h\left(\varDelta_{x,j}^{-1}\left(n^{-1/2}u + v\right)\right)}{\varsigma_{dx}\left(\varDelta_{x,j}^{-1}\left(n^{-1/2}u + v\right)\right)} + \frac{\delta}{\underline{\varsigma}_{dx}} \right) \rho_{d'x,j}\left(n^{-1/2}u + v\right) \mathrm{d}u \\
\leq \; & \int_{\sqrt{n}\left(\varDelta_{x,j}\left(\epsilon_{x,j-1}\right) - v\right)}^{\sqrt{n}\left(\varDelta_{x,j}\left(\epsilon_{x,j}\right) - v\right)} \mathbb{1}\left(u \leq -\frac{h\left(\varDelta_{x,j}^{-1}\left(v\right)\right)}{\varsigma_{dx}\left(\varDelta_{x,j}^{-1}\left(v\right)\right)} + \left| \frac{h\left(\varDelta_{x,j}^{-1}\left(n^{-1/2}u + v\right)\right)}{\varsigma_{dx}\left(\varDelta_{x,j}^{-1}\left(n^{-1/2}u + v\right)\right)} - \frac{h\left(\varDelta_{x,j}^{-1}\left(v\right)\right)}{\varsigma_{dx}\left(\varDelta_{x,j}^{-1}\left(v\right)\right)} \right| + \frac{\delta}{\underline{\varsigma}_{dx}} \right) \\
& \times \mathbb{1}\left(u \geq -\frac{h\left(\varDelta_{x,j}^{-1}\left(v\right)\right)}{\varsigma_{dx}\left(\varDelta_{x,j}^{-1}\left(v\right)\right)} - \left| \frac{h\left(\varDelta_{x,j}^{-1}\left(n^{-1/2}u + v\right)\right)}{\varsigma_{dx}\left(\varDelta_{x,j}^{-1}\left(n^{-1/2}u + v\right)\right)} - \frac{h\left(\varDelta_{x,j}^{-1}\left(v\right)\right)}{\varsigma_{dx}\left(\varDelta_{x,j}^{-1}\left(v\right)\right)} \right| - \frac{\delta}{\underline{\varsigma}_{dx}} \right)
\end{aligned}$$

$$\times \mathbb{1}\left(|u| \leq \frac{M+\delta}{\underline{\varsigma}_{dx}}\right) \rho_{d'x,j}\left(n^{-1/2}u+v\right) \mathrm{d}u$$

$$\leq \left(\frac{2\delta}{\underline{\varsigma}_{dx}} + 2 \sup_{u \in \mathscr{U}_n(v)} \left| \frac{h\left(\Delta_{x,j}^{-1}\left(n^{-1/2}u+v\right)\right)}{\varsigma_{dx}\left(\Delta_{x,j}^{-1}\left(n^{-1/2}u+v\right)\right)} - \frac{h\left(\Delta_{x,j}^{-1}(v)\right)}{\varsigma_{dx}\left(\Delta_{x,j}^{-1}(v)\right)} \right| \right) \sup_{u \in \mathscr{U}_n(v)} \rho_{d'x,j}\left(n^{-1/2}u+v\right), \quad \text{(A14)}$$

where

$$\mathscr{U}_n(v) \coloneqq \left[\sqrt{n}\left(\Delta_{x,j}\left(\epsilon_{x,j-1}\right)-v\right) \vee -\left(\frac{M+\delta}{\underline{\varsigma}_{dx}}\right), \sqrt{n}\left(\Delta_{x,j}\left(\epsilon_{x,j}\right)-v\right) \wedge \left(\frac{M+\delta}{\underline{\varsigma}_{dx}}\right)\right].$$

Since K is a compact subset of $C\left[\underline{\epsilon}_x, \overline{\epsilon}_x\right]$, K is uniformly equicontinuous (Dudley, 2002, Theorem 2.4.7): for all $\varepsilon > 0$, there exists some $\kappa > 0$ such that $\sup_{h \in K, |x-y| \leq \kappa} |h(x) - h(y)| < \varepsilon$. Since K is uniformly equicontinuous and ς_{dx} is continuous and bounded away from zero,

$$\lim_{n \uparrow \infty} \sup_{(v,h) \in [\underline{v}_x, \overline{v}_x] \times K} \left\{ \sup_{u \in \mathscr{U}_n(v)} \left| \frac{h\left(\Delta_{x,j}^{-1}\left(n^{-1/2}u+v\right)\right)}{\varsigma_{dx}\left(\Delta_{x,j}^{-1}\left(n^{-1/2}u+v\right)\right)} - \frac{h\left(\Delta_{x,j}^{-1}(v)\right)}{\varsigma_{dx}\left(\Delta_{x,j}^{-1}(v)\right)} \right| \right\} = 0. \quad \text{(A15)}$$

It is also easy to see that $\sup_{u \in \mathscr{U}_n(v)} \rho_{d'x,j}\left(n^{-1/2}u+v\right)$ is bounded uniformly in $v \in [\underline{v}_x, \overline{v}_x]$, when n is sufficiently large. (A12) follows from this result, (A13), (A14) and (A15).

Next, we show that (A4), (A8), (A12) and the result that $T_n(\delta) = o_p(1)$ for all $\delta > 0$ imply (A2). By (A12), there exists some $\delta_0 > 0$ such that

$$\sup_{(v,h) \in [\underline{v}_x, \overline{v}_x] \times K} \sqrt{n} \cdot \mathbb{P}\Gamma_{dx}(\cdot \mid v, \delta_0, h) < \frac{\varepsilon}{4}, \quad \text{(A16)}$$

for all sufficiently large n. We have

$$\begin{aligned}
&\Pr\left[\sup_{v \in [\underline{v}_x, \overline{v}_x]} \left|\mathbb{G}_n \gamma_{dx}\left(\cdot \mid v, \sqrt{n} \cdot H_x, 0\right)\right| > \varepsilon\right] \\
\leq\ &\Pr\left[\sup_{h \in \bigcup_{k=1}^{p_{\delta_0}} B(h_k, \delta_0)} \sup_{v \in [\underline{v}_x, \overline{v}_x]} \left|\mathbb{G}_n \gamma_{dx}(\cdot \mid v, h, 0)\right| > \varepsilon\right] + \Pr\left[\sqrt{n} \cdot H_x \notin \bigcup_{k=1}^{p_{\delta_0}} B(h_k, \delta_0)\right] \\
\leq\ &\Pr\left[\sup_{(v,h) \in [\underline{v}_x, \overline{v}_x] \times K} 2\sqrt{n} \cdot \mathbb{P}\Gamma_{dx}(\cdot \mid v, \delta_0, h) + T_n(\delta_0) > \varepsilon\right] + \frac{\varepsilon}{2} \\
\leq\ &\Pr\left[T_n(\delta_0) > \frac{\varepsilon}{2}\right] + \frac{\varepsilon}{2},
\end{aligned}$$

for all sufficiently large n, where the first inequality follows from the union bound, the second inequality follows from (A4) and (A8), and the third inequality follows from (A16). The conclusion follows from this result and the result that $\Pr\left[T_n(\delta_0) > \varepsilon/2\right] \leq \varepsilon/2$ for all sufficiently large n. ■

The following lemma shows the Hadamard differentiability of Ψ_{dx} (at Δ_x) and derives the form of the Hadamard derivative (at Δ_x).

Lemma 2. *Under the assumptions in the statement of Theorem 1, Ψ_{dx} is Hadamard differentiable at Δ_x tangentially to $C[\underline{\epsilon}_x, \overline{\epsilon}_x]$:*

$$\lim_{n\uparrow\infty}\left\|\frac{\Psi_{dx}\left(\Delta_x+t_nh_n\right)-\Psi_{dx}\Delta_x}{t_n}-\psi_{dx}h_0\right\|_\infty=0, \tag{A17}$$

where $\{(h_n,t_n)\}_{n=1}^\infty$ is a sequence in $\ell^\infty[\underline{\epsilon}_x, \overline{\epsilon}_x]\times\mathbb{R}$ that converges to $(h_0,0)\in C[\underline{\epsilon}_x, \overline{\epsilon}_x]\times\mathbb{R}$ and $\psi_{dx}:\ell^\infty[\underline{\epsilon}_x, \overline{\epsilon}_x]\to\ell^\infty[\underline{v}_x,\overline{v}_x]$ is a continuous linear operator defined by

$$\psi_{dx}h(v):=-\sum_{j=1}^m\left|\rho_{d'x,j}(v)\right|h\left(\Delta_{x,j}^{-1}(v)\right),\ v\in[\underline{v}_x,\overline{v}_x],$$

for all $h\in\ell^\infty[\underline{\epsilon}_x, \overline{\epsilon}_x]$.

Proof of Lemma 2. By change of variables $u=\left(\Delta_{x,j}(e)-v\right)/t_n$,

$$\begin{aligned}
&\frac{\Psi_{dx}\left(\Delta_x+t_nh_n\right)-\Psi_{dx}\Delta_x}{t_n}\\
=\ &\frac{1}{t_n}\sum_{j=1}^m\int_{\epsilon_{x,j-1}}^{\epsilon_{x,j}}\left\{\mathbb{1}\left(\Delta_{x,j}(e)+t_nh_n(e)\le v\right)-\mathbb{1}\left(\Delta_{x,j}(e)\le v\right)\right\}f_{(\epsilon,D)|X}\left(e,d'\mid x\right)\mathrm{d}e\\
=\ &\sum_{j=1}^m\int_{\frac{\Delta_{x,j}\left(\epsilon_{x,j-1}\right)-v}{t_n}}^{\frac{\Delta_{x,j}\left(\epsilon_{x,j}\right)-v}{t_n}}\left\{\mathbb{1}\left(u\le-h_n\left(\Delta_{x,j}^{-1}(t_nu+v)\right)\right)-\mathbb{1}(u\le0)\right\}\rho_{d'x,j}(t_nu+v)\,\mathrm{d}u.
\end{aligned}\tag{A18}$$

Assume that $\Delta_{x,j}$ is strictly increasing without loss of generality. Then, by (A6) and the triangle inequality,

$$\begin{aligned}
&\left|\int_{\frac{\Delta_j\left(\epsilon_{j-1}\right)-v}{t_n}}^{\frac{\Delta_j\left(\epsilon_j\right)-v}{t_n}}\left\{\mathbb{1}\left(u\le-h_n\left(\Delta_j^{-1}(t_nu+v)\right)\right)-\mathbb{1}\left(u\le-h_0\left(\Delta_j^{-1}(t_nu+v)\right)\right)\right\}\rho_{d'x,j}(t_nu+v)\,\mathrm{d}u\right|\\
\le\ &\int_{\frac{\Delta_{x,j}\left(\epsilon_{x,j-1}\right)-v}{t_n}}^{\frac{\Delta_{x,j}\left(\epsilon_{x,j}\right)-v}{t_n}}\mathbb{1}\left(|u|\le\|h_0\|_\infty+\|h_n-h_0\|_\infty\right)\\
&\times\mathbb{1}\left(u>-h_0\left(\Delta_{x,j}^{-1}(v)\right)-\left|h_0\left(\Delta_{x,j}^{-1}(t_nu+v)\right)-h_0\left(\Delta_{x,j}^{-1}(v)\right)\right|-\|h_n-h_0\|_\infty\right)\\
&\times\mathbb{1}\left(u\le-h_0\left(\Delta_{x,j}^{-1}(v)\right)+\left|h_0\left(\Delta_{x,j}^{-1}(t_nu+v)\right)-h_0\left(\Delta_{x,j}^{-1}(v)\right)\right|+\|h_n-h_0\|_\infty\right)\\
&\times\rho_{d'x,j}(t_nu+v)\,\mathrm{d}u\\
\le\ &2\left\{\|h_n-h_0\|_\infty+\sup_{u\in\mathscr{U}_n'(v)}\left|h_0\left(\Delta_{x,j}^{-1}(t_nu+v)\right)-h_0\left(\Delta_{x,j}^{-1}(v)\right)\right|\right\}\left\{\sup_{u\in\mathscr{U}_n'(v)}\rho_{d'x,j}(t_nu+v)\right\},
\end{aligned}\tag{A19}$$

where

$$\mathscr{U}_n'(v):=\left[\frac{\Delta_{x,j}\left(\epsilon_{x,j-1}\right)-v}{t_n}\vee\left(-\|h_0\|_\infty-\|h_n-h_0\|_\infty\right),\frac{\Delta_{x,j}\left(\epsilon_{x,j}\right)-v}{t_n}\wedge\left(\|h_0\|_\infty+\|h_n-h_0\|_\infty\right)\right].$$

It follows from the uniform continuity of h_0 that

$$\lim_{n\uparrow\infty}\sup_{v\in[\underline{v}_x,\overline{v}_x]}\sup_{u\in\mathscr{U}_n'(v)}\left|h_0\left(\Delta_{x,j}^{-1}(t_nu+v)\right)-h_0\left(\Delta_{x,j}^{-1}(v)\right)\right|=0. \tag{A20}$$

Similarly,

$$\begin{aligned}
&\left|\int_{\frac{\Delta_{x,j}\left(\epsilon_{x,j-1}\right)-v}{t_n}}^{\frac{\Delta_{x,j}\left(\epsilon_{x,j}\right)-v}{t_n}}\left\{\mathbb{1}\left(u\leq -h_0\left(\Delta_{x,j}^{-1}(t_nu+v)\right)\right)-\mathbb{1}\left(u\leq -h_0\left(\Delta_{x,j}^{-1}(v)\right)\right)\right\}\rho_{d'x,j}(t_nu+v)\,\mathrm{d}u\right| \\
\leq\;& \int_{\frac{\Delta_{x,j}\left(\epsilon_{x,j-1}\right)-v}{t_n}}^{\frac{\Delta_{x,j}\left(\epsilon_{x,j}\right)-v}{t_n}}\mathbb{1}\left(|u|\leq\|h_0\|_\infty\right)\mathbb{1}\left(u\leq -h_0\left(\Delta_{x,j}^{-1}(v)\right)+\left|h_0\left(\Delta_{x,j}^{-1}(t_nu+v)\right)-h_0\left(\Delta_{x,j}^{-1}(v)\right)\right|\right) \\
&\times\mathbb{1}\left(u> -h_0\left(\Delta_{x,j}^{-1}(v)\right)-\left|h_0\left(\Delta_{x,j}^{-1}(t_nu+v)\right)-h_0\left(\Delta_{x,j}^{-1}(v)\right)\right|\right)\rho_{d'x,j}(t_nu+v)\,\mathrm{d}u \\
\leq\;& 2\left\{\sup_{u\in\mathscr{U}_n'(v)}\left|h_0\left(\Delta_{x,j}^{-1}(t_nu+v)\right)-h_0\left(\Delta_{x,j}^{-1}(v)\right)\right|\right\}\left\{\sup_{u\in\mathscr{U}_n'(v)}\rho_{d'x,j}(t_nu+v)\right\}.
\end{aligned} \tag{A21}$$

By (A20) and the fact that $\sup_{u\in\mathscr{U}_n'(v)}\rho_{d'x,j}\left(n^{-1/2}u+v\right)$ is bounded uniformly in $v\in[\underline{v}_x,\overline{v}_x]$ for all sufficiently large n, the right hand sides of the second inequalities in (A19) and (A21) are both $o(1)$, uniformly in $v\in[\underline{v}_x,\overline{v}_x]$.

Therefore, by the calculations above and (A18),

$$\begin{aligned}
&\frac{\Psi_{dx}(\Delta_x+t_nh_n)-\Psi_{dx}\Delta_x}{t_n} \\
=\;& \sum_{j=1}^{m}\int_{\frac{\Delta_{x,j}\left(\epsilon_{x,j-1}\right)-v}{t_n}}^{\frac{\Delta_{x,j}\left(\epsilon_{x,j}\right)-v}{t_n}}\left\{\mathbb{1}\left(u\leq -h_0\left(\Delta_{x,j}^{-1}(v)\right)\right)-\mathbb{1}(u\leq 0)\right\}\rho_{d'x,j}(t_nu+v)\,\mathrm{d}u+o(1),
\end{aligned} \tag{A22}$$

uniformly in $v\in[\underline{v}_x,\overline{v}_x]$. By (A6) and the triangle inequality,

$$\begin{aligned}
&\left|\int_{\frac{\Delta_{x,j}\left(\epsilon_{x,j-1}\right)-v}{t_n}}^{\frac{\Delta_{x,j}\left(\epsilon_{x,j}\right)-v}{t_n}}\left\{\mathbb{1}\left(u\leq -h_0\left(\Delta_{x,j}^{-1}(v)\right)\right)-\mathbb{1}(u\leq 0)\right\}\left\{\rho_{d'x,j}(t_nu+v)-\rho_{d'x,j}(v)\right\}\mathrm{d}u\right| \\
\leq\;& \int_{\frac{\Delta_{x,j}\left(\epsilon_{x,j-1}\right)-v}{t_n}}^{\frac{\Delta_{x,j}\left(\epsilon_{x,j}\right)-v}{t_n}}\mathbb{1}\left(|u|\leq\|h_0\|_\infty\right)\left|\rho_{d'x,j}(t_nu+v)-\rho_{d'x,j}(v)\right|\mathrm{d}u \\
\leq\;& 2\|h_0\|_\infty\left\{\sup_{u\in\mathscr{U}_n'(v)}\left|\rho_{d'x,j}(t_nu+v)-\rho_{d'x,j}(v)\right|\right\} \\
=\;& o(1),
\end{aligned}$$

uniformly in $v\in[\underline{v}_x,\overline{v}_x]$. By this result and (A22),

$$\frac{\Psi_{dx}(\Delta_x+t_nh_n)-\Psi_{dx}\Delta_x}{t_n}$$

$$= \sum_{j=1}^{m} \rho_{d'x,j}(v) \int_{\frac{\Delta_{x,j}(\epsilon_{x,j-1})-v}{t_n}}^{\frac{\Delta_{x,j}(\epsilon_{x,j})-v}{t_n}} \left\{ \mathbb{1}\left(u \leq -h_0\left(\Delta_{x,j}^{-1}(v)\right)\right) - \mathbb{1}(u \leq 0) \right\} \mathrm{d}u + o(1),$$

uniformly in $v \in [\underline{v}_x, \overline{v}_x]$. The conclusion follows from this result and (A6). ■

Let

$$\begin{aligned} R_{d'x}(y) &:= \Pr[Y \leq \phi_{dx}(y), D = d \mid X = x] + \Pr[Y \leq y, D = d' \mid X = x] \\ L_{dx}(W_i, y) &:= \frac{\mathbb{1}(Y_i \leq \phi_{dx}(y), D_i = d) + \mathbb{1}(Y_i \leq y, D_i = d') - R_{d'x}(y)}{\zeta_{dx}(\phi_{dx}(y))} \cdot \pi_x(Z_i). \end{aligned}$$

It is easy to see that we have

$$L_{dx}\left(W_j, g\left(d', x, \epsilon_i\right)\right) = \frac{\mathbb{1}(\epsilon_j \leq \epsilon_i) - F_{\epsilon|X}(\epsilon_i \mid x)}{\zeta_{dx}(g(d, x, \epsilon_i))} \cdot \pi_x(Z_j). \tag{A23}$$

Let $\widehat{\phi}_{dx}(y)$ denote the leave-in version of $\widehat{\phi}_{dx}^{(-i)}(y)$ that minimizes the sample analogue of (5), which is defined by the right hand side of (6) with all ranges of summation changed to $\sum_{j=1}^{n}$. Under the assumptions in the statement of Theorem 1, by Lemma 2 of MMY , we have the Bahadur-type representation result

$$\widehat{\phi}_{dx}(y) - \phi_{dx}(y) = \frac{1}{n} \sum_{i=1}^{n} L_{dx}(W_i, y) + \xi_{dx}(y), \tag{A24}$$

where $\xi_{dx}(y) = O_p\left((\log(n)/n)^{3/4}\right)$, uniformly in $y \in \left[\underline{y}_{d'x}, \overline{y}_{d'x}\right]$. Lemma 2 of MMY shows that $\|\xi_{dx}\|_{\infty} \leq r_n$ wpa1 for some deterministic sequence $\{r_n\}_{n=1}^{\infty}$ which is proportional to $(\log(n)/n)^{3/4}$. It is clear from the proof of Lemma 2 of MMY that the linearization result (A24) also holds for $\widehat{\phi}_{dx}^{(-i)}(y)$ and the remainder term denoted by $\xi_{dx}^{(-i)}(y)$ has the same order of magnitude, uniformly in $i = 1, ..., n$.

By using (A23) and (A24), we can write

$$\begin{aligned} \widehat{\Delta}_i - \Delta_i &= D_i\left(\phi_{0x}(Y_i) - \widehat{\phi}_{0x}^{(-i)}(Y_i)\right) + (1 - D_i)\left(\widehat{\phi}_{1x}^{(-i)}(Y_i) - \phi_{1x}(Y_i)\right) \\ &= -D_i\left\{\frac{1}{n}\sum_{j=1}^{n} L_{0x}(W_j, g(1, x, \epsilon_i))\right\} + (1 - D_i)\left\{\frac{1}{n}\sum_{j=1}^{n} L_{1x}(W_j, g(0, x, \epsilon_i))\right\} + \xi_i, \\ &= \left\{\frac{D_i}{\varsigma_{0x}(\epsilon_i)} + \frac{(1 - D_i)}{\varsigma_{1x}(\epsilon_i)}\right\} H_x(\epsilon_i) + \xi_i, \end{aligned} \tag{A25}$$

where $\xi_i := -D_i \xi_{0x}^{(-i)}(Y_i) + (1 - D_i)\xi_{1x}^{(-i)}(Y_i)$.

Now by (A25), we can write

$$\widehat{F}_{\Delta|X}(v \mid x) - \breve{F}_{\Delta|X}(v \mid x)$$

$$= \frac{1}{n}\sum_{i=1}^{n}\sum_{d\in\{0,1\}}\left(\mathbb{1}\left(\Delta_i+\frac{H_x(\epsilon_i)}{\varsigma_{dx}(\epsilon_i)}+\xi_i\le v\right)-\mathbb{1}\left(\Delta_i+\frac{H_x(\epsilon_i)}{\varsigma_{dx}(\epsilon_i)}\le v\right)\right)\mathbb{1}\left(D_i=d'\right). \quad \text{(A26)}$$

Using these results and notations, we can show the following lemma.

Lemma 3. *Under the assumptions in the statement of Theorem 1, (14) holds.*

Proof of Lemma 3. Let $\bar{\xi}_n \coloneqq \max\{|\xi_1|,...,|\xi_n|\}$. It follows from (A24) that $\bar{\xi}_n \le r_n$ wpa1, where $r_n = O\left((\log(n)/n)^{3/4}\right)$. By this result, (A6), (A26) and the triangle inequality, we have

$$\begin{aligned}
&\left|\sqrt{n}\cdot\left(\widehat{F}_{\Delta|X}(v\mid x)-\breve{F}_{\Delta|X}(v\mid x)\right)\right| \\
\le\ & \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{d\in\{0,1\}}\mathbb{1}\left(v-\bar{\xi}_n<\Delta_i+\frac{H_x(\epsilon_i)}{\varsigma_{dx}(\epsilon_i)}\le v+\bar{\xi}_n\right)\mathbb{1}\left(D_i=d'\right) \\
\le\ & \sqrt{n}\left(\breve{F}_{\Delta|X}(v+r_n\mid x)-\breve{F}_{\Delta|X}(v\mid x)\right)+\sqrt{n}\left(\breve{F}_{\Delta|X}(v\mid x)-\breve{F}_{\Delta|X}(v-r_n\mid x)\right), \quad \text{(A27)}
\end{aligned}$$

where the second inequality holds wpa1.

Write

$$\begin{aligned}
&\breve{F}_{\Delta|X}(v+r_n\mid x)-\breve{F}_{\Delta|X}(v\mid x)= \\
&\qquad\left\{\breve{F}_{\Delta|X}(v+r_n\mid x)-\widetilde{F}_{\Delta|X}(v\mid x)\right\}-\left\{\breve{F}_{\Delta|X}(v\mid x)-\widetilde{F}_{\Delta|X}(v\mid x)\right\}. \quad \text{(A28)}
\end{aligned}$$

By Lemma 1 and the functional delta method,

$$\breve{F}_{\Delta|X}(v\mid x)-\widetilde{F}_{\Delta|X}(v\mid x)=\sum_{d\in\{0,1\}}\psi_{dx}\left(\frac{\sqrt{n}\cdot H_x}{\varsigma_{dx}}\right)(v)+o_p(1), \quad \text{(A29)}$$

uniformly in $v\in[\underline{v}_x,\overline{v}_x]$. By Slutsky's theorem (Kosorok, 2007, Theorem 7.15(ii)), Lemma 2 and the functional delta method,

$$\begin{aligned}
\sqrt{n}\left(\breve{F}_{\Delta|X}(v+r_n\mid x)-\widetilde{F}_{\Delta|X}(v\mid x)\right) &= \sqrt{n}\sum_{d\in\{0,1\}}\left\{\Psi_{dx}\left(\Delta_x+\frac{H_x}{\varsigma_{dx}}-r_n\right)-\Psi_{dx}\Delta_x\right\}(v)+o_p(1), \\
&= \sum_{d\in\{0,1\}}\psi_{dx}\left(\frac{\sqrt{n}\cdot H_x}{\varsigma_{dx}}-\sqrt{n}\cdot r_n\right)(v)+o_p(1), \quad \text{(A30)}
\end{aligned}$$

uniformly in $v\in[\underline{v}_x,\overline{v}_x]$, where the first equality follows from the same arguments as those used to prove Lemma 1 and the second equality follows from the functional delta method. It now follows from the linearity of ψ_{dx}, (A28), (A29), (A30) and $r_n = O\left((\log(n)/n)^{3/4}\right)$ that the first term on the right hand side of the second inequality in (A27) is $o_p(1)$, uniformly in $v\in[\underline{v}_x,\overline{v}_x]$. By using similar arguments, we can show that the second term on the right hand side of the second inequality in (A27) also is $o_p(1)$, uniformly in $v\in[\underline{v}_x,\overline{v}_x]$. The conclusion of the lemma follows from these results and (A27). ■

Now we show that the conclusion in Theorem 1 follows from Lemmas 1, 2 and 3.

Proof of Theorem 1. By Lemma 3 and (A29), we have

$$\begin{aligned}\sqrt{n}\left(\widehat{F}_{\Delta|X}(v\mid x)-\widetilde{F}_{\Delta|X}(v\mid x)\right)&=\sqrt{n}\sum_{j=1}^{m}\omega_{x,j}(v)\,H_x\left(\Delta_{x,j}^{-1}(v)\right)+o_p(1)\\&=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{j=1}^{m}\omega_{x,j}(v)\left\{\mathbb{1}\left(\epsilon_i\le\Delta_{x,j}^{-1}(v)\right)-F_{\epsilon|X}\left(\Delta_{x,j}^{-1}(v)\mid x\right)\right\}\pi_x(Z_i)+o_p(1),\quad\text{(A31)}\end{aligned}$$

uniformly in $v\in[\underline{v}_x,\overline{v}_x]$, and therefore

$$\begin{aligned}S_F(v\mid x)&=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\Bigg\{\left(\mathbb{1}\left(\Delta_x(\epsilon_i)\le v\right)-F_{\Delta|X}(v\mid x)\right)\\&\quad+\sum_{j=1}^{m}\omega_{x,j}(v)\left(\mathbb{1}\left(\epsilon_i\le\Delta_{x,j}^{-1}(v)\right)-F_{\epsilon|X}\left(\Delta_{x,j}^{-1}(v)\mid x\right)\right)\pi_x(Z_i)\Bigg\}+o_p(1)\quad\text{(A32)}\end{aligned}$$

uniformly in $v\in[\underline{v}_x,\overline{v}_x]$. Let

$$\iota_x(U_i\mid v):=\mathbb{1}\left(\Delta_x(\epsilon_i)\le v\right)+\sum_{j=1}^{m}\omega_{x,j}(v)\left(\mathbb{1}\left(\epsilon_i\le\Delta_{x,j}^{-1}(v)\right)-F_{\epsilon|X}\left(\Delta_{x,j}^{-1}(v)\mid x\right)\right)\pi_x(Z_i)$$

and the leading terms on the right hand side of (A32) can be written as $\mathbb{G}_n\iota_x(\cdot\mid v)$. It clear that since $\mathrm{E}\left[\pi_x(Z)\mid X=x\right]=0$ and Z and ϵ are conditionally independent given $X=x$, the two leading terms on the right hand side of (A32) are uncorrelated. By Kosorok (2007, Lemma 9.7, (iii), (iv) and (vi)),

$$\left\{e\mapsto\sum_{j=1}^{m}\omega_{x,j}(v)\left(\mathbb{1}\left(e\le\Delta_{x,j}^{-1}(v)\right)-F_{\epsilon|X}\left(\Delta_{x,j}^{-1}(v)\mid x\right)\right):v\in[\underline{v}_x,\overline{v}_x]\right\}$$

and $\{e\mapsto\mathbb{1}\left(\Delta_x(e)\le v\right):v\in[\underline{v}_x,\overline{v}_x]\}$ are both VC classes of functions. By Kosorok (2007, Theorem 8.19), they are also Donsker classes. By this result, (A32), Kosorok (2007, Theorem 9.30(i)), Kosorok (2007, Corollary 9.32(i) and (v)), Kosorok (2007, Lemma 7.23(i)) and also the fact that $\mathrm{E}\left[\pi_x(Z)\mid X=x\right]=0$ and Z and ϵ are conditionally independent given $X=x$, $S_F(\cdot\mid x)$ weakly converges to a tight Gaussian random element in $\ell^\infty[\underline{v}_x,\overline{v}_x]$ with zero mean and the covariance structure given by

$$\begin{aligned}&\left(F_{\Delta|X}\left(v\wedge v'\mid x\right)-F_{\Delta|X}(v\mid x)\,F_{\Delta|X}\left(v'\mid x\right)\right)+\left(p_{1|x}^{-1}+p_{0|x}^{-1}\right)\\&\times\sum_{j=1}^{m}\sum_{k=1}^{m}\omega_{x,j}(v)\,\omega_{x,k}\left(v'\right)\left(F_{\epsilon|X}\left(\Delta_{x,j}^{-1}(v)\wedge\Delta_{x,j}^{-1}\left(v'\right)\mid x\right)-F_{\epsilon|X}\left(\Delta_{x,j}^{-1}(v)\mid x\right)F_{\epsilon|X}\left(\Delta_{x,j}^{-1}\left(v'\right)\mid x\right)\right),\end{aligned}$$

where $v, v' \in [\underline{v}_x, \overline{v}_x]$, which has the same distribution as $\mathbb{F}(\cdot \mid x)$ (Kosorok, 2007, Lemma 7.3). Then the conclusion in Part (i) follows immediately. The conclusion in Part (ii) follows from Part (i) and the CMT. ■

Proof of Corollary 1. For any $\tau \in (0,1)$ and sufficiently small $\varepsilon > 0$ such that $Q_{\Delta|X}(\tau \mid x) + \varepsilon$ is an interior point of $\mathscr{S}_{\Delta|X=x}$, we have

$$\begin{aligned}&\Pr\left[\widehat{Q}_{\Delta|X}(\tau \mid x) \leq Q_{\Delta|X}(\tau \mid x) + \varepsilon\right] \geq \\ &\Pr\left[\left|\widehat{F}_{\Delta|X}\left(Q_{\Delta|X}(\tau \mid x) + \varepsilon \mid x\right) - F_{\Delta|X}\left(Q_{\Delta|X}(\tau \mid x) + \varepsilon \mid x\right)\right| \leq F_{\Delta|X}\left(Q_{\Delta|X}(\tau \mid x) + \varepsilon \mid x\right) - \tau\right],\end{aligned} \tag{A33}$$

where the inequality follows from Van der Vaart (2000, Lemma 21.2(i)) and by Theorem 1, the right hand side of the inequality converges to one. By similar arguments, for all sufficiently small $\varepsilon > 0$ such that $Q_{\Delta|X}(\tau \mid x) - \varepsilon$ is an interior point of $\mathscr{S}_{\Delta|X=x}$,

$$\Pr\left[\widehat{Q}_{\Delta|X}(\tau \mid x) \geq Q_{\Delta|X}(\tau \mid x) - \varepsilon\right] \to 1,$$

as $n \uparrow \infty$. It follows that $\widehat{Q}_{\Delta|X}(\tau \mid x) \to_p Q_{\Delta|X}(\tau \mid x)$ for all $\tau \in (0,1)$.

Now fix some small $\varepsilon > 0$ such that $[a,b] \coloneqq \left[Q_{\Delta|X}(\underline{\tau} \mid x) - \varepsilon, Q_{\Delta|X}(\overline{\tau} \mid x) + \varepsilon\right]$ is an inner closed sub-interval of $\mathscr{S}_{\Delta|X=x}$. It follows from the consistency of $\widehat{Q}_{\Delta|X}(\underline{\tau} \mid x)$ and $\widehat{Q}_{\Delta|X}(\overline{\tau} \mid x)$ that wpa1, $\widehat{Q}_{\Delta|X}(\underline{\tau} \mid x) > a$ and $\widehat{Q}_{\Delta|X}(\overline{\tau} \mid x) < b$. By Van der Vaart (2000, Lemma 21.2(ii,iii)), on this event, for all $\tau \in [\underline{\tau}, \overline{\tau}]$, $\widehat{Q}_{\Delta|X}(\tau \mid x)$ can be written as the image of the restriction $\widehat{F}_{\Delta|X}(\cdot \mid x)\big|_{[a,b]}$ under the map ϕ_τ defined in the statement of Van der Vaart (2000, Lemma 21.3). Clearly, $Q_{\Delta|X}(\tau \mid x)$ is the image of the restriction $F_{\Delta|X}(\cdot \mid x)\big|_{[a,b]}$ under the same map. Let $\dot{F}_{\Delta|X}(\cdot \mid x)$ be a modification of $\widehat{F}_{\Delta|X}(\cdot \mid x)$ defined in the following way. Let $\dot{F}_{\Delta|X}(v \mid x)$ be $\widehat{F}_{\Delta|X}(v \mid x)$ for all $v \in \mathbb{R}$ if $\widehat{Q}_{\Delta|X}(\underline{\tau} \mid x) > a$ and $\widehat{Q}_{\Delta|X}(\overline{\tau} \mid x) < b$ and let $\dot{F}_{\Delta|X}(v \mid x)$ be $F_{\Delta|X}(v \mid x)$ for all $v \in \mathbb{R}$ otherwise. Then, $\widehat{Q}_{\Delta|X}(\tau \mid x) - \phi_\tau\left(\dot{F}_{\Delta|X}(\cdot \mid x)\big|_{[a,b]}\right) = 0$ for all $\tau \in [\underline{\tau}, \overline{\tau}]$ wpa1 and therefore

$$\widehat{Q}_{\Delta|X}(\tau \mid x) - \phi_\tau\left(\dot{F}_{\Delta|X}(\cdot \mid x)\big|_{[a,b]}\right) = o_p(\alpha_n), \tag{A34}$$

uniformly in $\tau \in [\underline{\tau}, \overline{\tau}]$, for any $\alpha_n \downarrow 0$. Similarly, we also have

$$\widehat{F}_{\Delta|X}(v \mid x) - \dot{F}_{\Delta|X}(v \mid x) = o_p(\alpha_n), \tag{A35}$$

uniformly in $v \in [a,b]$, for any $\alpha_n \downarrow 0$. By the convergence in distribution of $S_F(\cdot \mid x)$ in $\ell^\infty[a,b]$ implied by Theorem 1(i) with $[\underline{v}_x, \overline{v}_x] = [a,b]$ and (A35), $\sqrt{n}\left(\dot{F}_{\Delta|X}(\cdot \mid x)\big|_{[a,b]} - F_{\Delta|X}(\cdot \mid x)\big|_{[a,b]}\right)$ converges in distribution in $\ell^\infty[a,b]$ to a tight Gaussian process. By this result, (A34), (A35), Van der Vaart (2000, Lemma 21.4(i)), and the functional delta method,

$$\begin{aligned} S_Q(\tau \mid x) &= \sqrt{n}\left\{\phi_\tau\left(\dot{F}_{\Delta|X}(\cdot \mid x)\Big|_{[a,b]}\right) - \phi_\tau\left(F_{\Delta|X}(\cdot \mid x)\Big|_{[a,b]}\right)\right\} + o_p^\dagger(1) \\ &= -\frac{S_F\left(Q_{\Delta|X}(\tau \mid x) \mid x\right)}{f_{\Delta|X}\left(Q_{\Delta|X}(\tau \mid x) \mid x\right)} + o_p(1), \end{aligned} \tag{A36}$$

uniformly in $\tau \in [\underline{\tau}, \overline{\tau}]$. The conclusion in Part (i) follows from this result and Theorem 1(i) and the CMT. Part (ii) follows from Part (i) and the CMT. ■

B Proofs of results in Section 4

We write $\xi_n^\dagger = o_p^\dagger(1)$ if for all $\varepsilon > 0$, $\Pr_\dagger\left[\left|\xi_n^\dagger\right| > \varepsilon\right] = o_p(1)$. It is easy to check that $\xi_n^\dagger = o_p^\dagger(1)$ if and only if for all $\varepsilon > 0$, $\Pr_\dagger\left[\left|\xi_n^\dagger\right| > \varepsilon\right] \leq \varepsilon$ wpa1. It essentially follows from the Markov's inequality that if $\xi_n = o_p(1)$, then we also have $\xi_n = o_p^\dagger(1)$: for all $\varepsilon, \delta > 0$,

$$\Pr\left[\Pr_\dagger\left[|\xi_n| > \varepsilon\right] > \delta\right] \leq \frac{\Pr\left[|\xi_n| > \varepsilon\right]}{\delta} \downarrow 0, \tag{B1}$$

as $n \uparrow \infty$. If ξ_n depends only on the original data, then $\Pr_\dagger\left[|\xi_n| > \varepsilon\right] = \mathbb{1}\left(|\xi_n| > \varepsilon\right)$ and $\mathbb{1}\left(|\xi_n| > \varepsilon\right) = 0$ wpa1. Moreover, it is also easy to see that properties of o_p carry over to $o_p^\dagger$ (e.g., $o_p^\dagger(\alpha_n) + o_p^\dagger(\beta_n) = o_p^\dagger(\beta_n)$ if $\alpha_n = O(\beta_n)$). Let $g^{-1}(d, x, \cdot)$ denote the inverse of $g(d, x, \cdot)$, $\epsilon_i^\dagger := D_i^\dagger g^{-1}\left(1, x, Y_i^\dagger\right) + \left(1 - D_i^\dagger\right) g^{-1}\left(0, x, Y_i^\dagger\right)$, $U_i^\dagger := \left(\epsilon_i^\dagger, D_i^\dagger, Z_i^\dagger\right)$ and $\Delta_i^\dagger := \Delta_x\left(\epsilon_i^\dagger\right)$. Denote $\mathbb{P}_n^\dagger := n^{-1}\sum_{i=1}^n f\left(U_i^\dagger\right)$ and $\mathbb{G}_n^\dagger := \sqrt{n}\left(\mathbb{P}_n^\dagger - \mathbb{P}_n\right)$.

Let

$$\begin{aligned} \widehat{F}_{\Delta|X}^\dagger(v \mid x) &:= \frac{1}{n}\sum_{i=1}^n \mathbb{1}\left(\widehat{\Delta}_i^\dagger \leq v\right) \\ \widetilde{F}_{\Delta|X}^\dagger(v \mid x) &:= \frac{1}{n}\sum_{i=1}^n \mathbb{1}\left(\Delta_i^\dagger \leq v\right) \\ \breve{F}_{\Delta|X}^\dagger(v \mid x) &:= \frac{1}{n}\sum_{i=1}^n \sum_{d \in \{0,1\}} \mathbb{1}\left(\Delta_x\left(\epsilon_i^\dagger\right) + \frac{H_x^\dagger\left(\epsilon_i^\dagger\right)}{\varsigma_{dx}\left(\epsilon_i^\dagger\right)} \leq v\right) \mathbb{1}\left(D_i^\dagger = d'\right), \end{aligned}$$

where $H_x^\dagger$ is the bootstrap analogue of H_x:

$$H_x^\dagger(e) := \frac{1}{n}\sum_{i=1}^n \left\{\mathbb{1}\left(\epsilon_i^\dagger \leq e\right) - F_{\epsilon|X}(e \mid x)\right\} \pi_x\left(Z_i^\dagger\right).$$

The following lemma is a bootstrap analogue of Lemma 1.

Lemma 4. *Under the assumptions in the statement of Theorem 2, we have*

$$\breve{F}^{\dagger}_{\Delta|X}(v \mid x) - \widetilde{F}^{\dagger}_{\Delta|X}(v \mid x) - \sum_{d\in\{0,1\}} \left\{ \Psi_{dx}\left(\Delta_x + \frac{H^{\dagger}_x}{\varsigma_{dx}}\right) - \Psi_{dx}\Delta_x \right\}(v) = o^{\dagger}_p\left(n^{-1/2}\right),$$

uniformly in $v \in [\underline{v}_x, \overline{v}_x]$

Proof of Lemma 4. We can write

$$\begin{aligned} & \breve{F}^{\dagger}_{\Delta|X}(v \mid x) - \widetilde{F}^{\dagger}_{\Delta|X}(v \mid x) - \sum_{d\in\{0,1\}} \left\{ \Psi_{dx}\left(\Delta_x + \frac{H^{\dagger}_x}{\varsigma_{dx}}\right) - \Psi_{dx}\Delta_x \right\}(v) \\ = \; & \frac{1}{\sqrt{n}} \cdot \left(\mathbb{G}^{\dagger}_n + \mathbb{G}_n\right) \gamma_x \left(\cdot \mid v, \sqrt{n} \cdot H^{\dagger}_x, 0\right). \end{aligned}$$

It suffices to show that for all $\varepsilon > 0$, the event

$$\Pr\nolimits_{\dagger}\left[\sup_{v\in[\underline{v}_x,\overline{v}_x]} \left|\left(\mathbb{G}^{\dagger}_n + \mathbb{G}_n\right)\gamma_{dx}\left(\cdot \mid v, \sqrt{n}\cdot H^{\dagger}_x, 0\right)\right| > \varepsilon\right] \leq \varepsilon \tag{B2}$$

occurs wpa1.

By Van der Vaart (2000, Theorem 23.7), $\sqrt{n}\left(H^{\dagger}_x - H_x\right) \rightsquigarrow_{\dagger} \mathbb{H}_x$ in $\ell^{\infty}[\underline{\epsilon}_x, \overline{\epsilon}_x]$ as $n \uparrow \infty$. Therefore for all $\varepsilon > 0$, the event

$$\sup_{h\in BL_1}\left|\mathrm{E}_{\dagger}\left[h\left(\sqrt{n}\left(H^{\dagger}_x - H_x\right)\right)\right] - \mathrm{E}\left[h\left(\mathbb{H}_x\right)\right]\right| < \varepsilon \tag{B3}$$

occurs wpa1. Now we show that for any open subset G of $\ell^{\infty}[\underline{\epsilon}_x, \overline{\epsilon}_x]$ and any $\varepsilon > 0$, we have

$$\Pr\nolimits_{\dagger}\left[\sqrt{n}\left(H^{\dagger}_x - H_x\right) \in G\right] \geq \Pr\left[\mathbb{H}_x \in G\right] - \varepsilon \tag{B4}$$

holds wpa1. To show (B4), first we note that there exists a sequence $\{f_m\}_{m=1}^{\infty}$ in BL_1 that converges pointwise to $\ell^{\infty}[\underline{\epsilon}_x, \overline{\epsilon}_x] \ni f \mapsto \mathbb{1}(f \in G)$ from below (see, e.g., the proof of Kosorok, 2007, Lemma 7.1 for one construction). By the monotone convergence theorem, $\lim_{m\uparrow\infty} \mathrm{E}[f_m(\mathbb{H}_x)] = \Pr[\mathbb{H}_x \in G]$. Then, by (B3), for any $\varepsilon > 0$,

$$\Pr\nolimits_{\dagger}\left[\sqrt{n}\left(H^{\dagger}_x - H_x\right) \in G\right] \geq \mathrm{E}\left[f_m\left(\sqrt{n}\left(H^{\dagger}_x - H_x\right)\right)\right] \geq \mathrm{E}\left[f_m\left(\mathbb{H}_x\right)\right] - \varepsilon, \forall m \in \mathbb{N},$$

holds wpa1. (B4) follows from these results.

Now fix some $\varepsilon > 0$ and also the compact set $K \subseteq C[\underline{\epsilon}_x, \overline{\epsilon}_x]$ such that $\Pr[\mathbb{H}_x \in K] > 1 - \varepsilon/4$. As in the proof of Lemma 1, for all $\delta > 0$, find $h_1, ..., h_{p_\delta} \in K$ such that $K^{\delta/2} \subseteq \bigcup_{k=1}^{p_\delta} B(h_k, \delta)$. By

(B4) with $G = K^{\delta/2}$, for each $\delta > 0$, the event

$$\Pr_{\dagger}\left[\sqrt{n}\left(H_x^{\dagger} - H_x\right) \in \bigcup_{k=1}^{p_\delta} B\left(h_k, \delta\right)\right] > 1 - \frac{\varepsilon}{2} \tag{B5}$$

occurs wpa1.

By the triangle inequality and (A7),

$$\begin{aligned}
& \sup_{h \in \bigcup_{k=1}^{p_\delta} B(h_k,\delta)} \sup_{v \in [\underline{v}_x, \overline{v}_x]} \left|\left(\mathbb{G}_n^{\dagger} + \mathbb{G}_n\right) \gamma_{dx}\left(\cdot \mid v, h + \sqrt{n} \cdot H_x, 0\right)\right| \\
\leq \; & \sup_{v \in [\underline{v}_x, \overline{v}_x]} \max_{k=1,\ldots,p_\delta} \sup_{h \in B(h_k,\delta)} \left|\left(\mathbb{G}_n^{\dagger} + \mathbb{G}_n\right) \gamma_{dx}\left(\cdot \mid v, h + \sqrt{n} \cdot H_x, 0\right)\right| \\
\leq \; & \sup_{v \in [\underline{v}_x, \overline{v}_x]} \max_{k=1,\ldots,p_\delta} \sup_{h \in B(h_k,\delta)} \left|\left(\mathbb{G}_n^{\dagger} + \mathbb{G}_n\right) \gamma_{dx}\left(\cdot \mid v, h + \sqrt{n} \cdot H_x, h_k + \sqrt{n} \cdot H_x\right)\right| \\
& + \sup_{v \in [\underline{v}_x, \overline{v}_x]} \max_{k=1,\ldots,p_\delta} \left|\left(\mathbb{G}_n^{\dagger} + \mathbb{G}_n\right) \gamma_{dx}\left(\cdot \mid v, h_k + \sqrt{n} \cdot H_x, 0\right)\right| \\
\leq \; & T_n^{\dagger}\left(\delta\right) + \mathit{\Gamma}_n\left(\delta\right),
\end{aligned} \tag{B6}$$

where

$$\begin{aligned}
T_n^{\dagger}\left(\delta\right) \; &:= \; \sup_{v \in [\underline{v}_x, \overline{v}_x]} \max_{k=1,\ldots,p_\delta} \left|\left(\mathbb{G}_n^{\dagger} + \mathbb{G}_n\right) \gamma_{dx}\left(\cdot \mid v, h_k + \sqrt{n} \cdot H_x, 0\right)\right| \\
& \quad + \sup_{v \in [\underline{v}_x, \overline{v}_x]} \max_{k=1,\ldots,p_\delta} \left|\left(\mathbb{G}_n^{\dagger} + \mathbb{G}_n\right) \mathit{\Gamma}_{dx}\left(\cdot \mid v, \delta, h_k + \sqrt{n} \cdot H_x\right)\right| \\
\mathit{\Gamma}_n\left(\delta\right) \; &:= \; \sup_{(v,h) \in [\underline{v}_x, \overline{v}_x] \times K} 2\sqrt{n} \cdot \mathbb{P}\mathit{\Gamma}_{dx}\left(\cdot \mid v, \delta, h + \sqrt{n} \cdot H_x\right).
\end{aligned} \tag{B7}$$

First we show that $T_n^{\dagger}(\delta) = o_p^{\dagger}(1)$ for all $\delta > 0$. It follows from the same arguments as those used to show (A2) that for all $h \in K$,

$$\mathbb{G}_n \gamma_{dx}\left(\cdot \mid v, h + \sqrt{n} \cdot H_x, 0\right) = o_p\left(1\right), \tag{B8}$$

uniformly in $[\underline{v}_x, \overline{v}_x]$. Let $\widehat{\sigma}_\gamma^2 := \sup_{v \in [\underline{v}_x, \overline{v}_x]} \mathbb{P}_n \gamma_{dx}^2(\cdot \mid v, h + \sqrt{n} \cdot H_x, 0)$. By using the same arguments as those used to show (A9), we have

$$\mathrm{E}_{\dagger}\left[\sup_{v \in [\underline{v}_x, \overline{v}_x]} \left|\mathbb{G}_n^{\dagger} \gamma_{dx}\left(\cdot \mid v, h, 0\right)\right|\right] \;\lesssim\; \widehat{\sigma}_\gamma \sqrt{\log\left(n\right)} + \frac{\log\left(n\right)}{\sqrt{n}} = o_p\left(1\right),$$

where the equality follows from (A7), $\sup_{h \in K} \|h\| \leq M$ and the fact that $\|\sqrt{n} \cdot H_x\|_\infty = O_p(1)$. Now it follows from this result, (B8) and (B1) that for all $\delta > 0$, the first term on the right hand side of the definition of $T_n^{\dagger}(\delta)$ in (B7) is $o_p^{\dagger}(1)$. By similar arguments, the second term on the right hand side of the definition of $T_n^{\dagger}(\delta)$ in (B7) is also $o_p^{\dagger}(1)$ for all $\delta > 0$.

Next we show that for all $\kappa > 0$, we can choose δ to be sufficiently small such that $\mathit{\Gamma}_n(\delta) \leq \kappa$

wpa1. We apply the same calculations (A13) and (A14) (with h replaced by $h+\sqrt{n}\cdot H_x$) for $\sqrt{n}\cdot\mathbb{P}\mathit{\Gamma}_{dx}\left(\cdot\mid v,\delta,h+\sqrt{n}\cdot H_x\right)$. The convergence in distribution of $\sqrt{n}\cdot H_x$ in $\ell^{\infty}\left[\underline{\epsilon}_x,\overline{\epsilon}_x\right]$ implies that $\sqrt{n}\cdot H_x$ is stochastically equicontinuous, i.e., for all $\kappa,\eta>0$, there exists some $\delta>0$ such that

$$\Pr\left[\sup_{x,y\in\left[\underline{\epsilon}_x,\overline{\epsilon}_x\right]:|x-y|<\delta}\left|\sqrt{n}\cdot H_x\left(x\right)-\sqrt{n}\cdot H_x\left(y\right)\right|>\kappa\right]<\eta,$$

for all large enough n. Then by this property, we have, for all $\kappa,\eta>0$,

$$\Pr\left[\sup_{v\in\left[\underline{v}_x,\overline{v}_x\right]}\left\{\sup_{u\in\mathscr{U}_n(v)}\left|\frac{\sqrt{n}\cdot H_x\left(\mathit{\Delta}_{x,j}^{-1}\left(n^{-1/2}u+v\right)\right)}{\varsigma_{dx}\left(\mathit{\Delta}_{x,j}^{-1}\left(n^{-1/2}u+v\right)\right)}-\frac{\sqrt{n}\cdot H_x\left(\mathit{\Delta}_{x,j}^{-1}\left(v\right)\right)}{\varsigma_{dx}\left(\mathit{\Delta}_{x,j}^{-1}\left(v\right)\right)}\right|\right\}>\kappa\right]<\eta,$$

for all large enough n. It follows from this result and (A15) that

$$\sup_{(v,h)\in\left[\underline{v}_x,\overline{v}_x\right]\times K}\left\{\sup_{u\in\mathscr{U}_n(v)}\left|\frac{\left(h+\sqrt{n}\cdot H_x\right)\left(\mathit{\Delta}_{x,j}^{-1}\left(n^{-1/2}u+v\right)\right)}{\varsigma_{dx}\left(\mathit{\Delta}_{x,j}^{-1}\left(n^{-1/2}u+v\right)\right)}-\frac{\left(h+\sqrt{n}\cdot H_x\right)\left(\mathit{\Delta}_{x,j}^{-1}\left(v\right)\right)}{\varsigma_{dx}\left(\mathit{\Delta}_{x,j}^{-1}\left(v\right)\right)}\right|\right\}=o_p\left(1\right),$$

and therefore, for any $\kappa>0$,

$$\sup_{(v,h)\in\left[\underline{v}_x,\overline{v}_x\right]\times K}\left\{\sup_{u\in\mathscr{U}_n(v)}\left|\frac{\left(h+\sqrt{n}\cdot H_x\right)\left(\mathit{\Delta}_{x,j}^{-1}\left(n^{-1/2}u+v\right)\right)}{\varsigma_{dx}\left(\mathit{\Delta}_{x,j}^{-1}\left(n^{-1/2}u+v\right)\right)}-\frac{\left(h+\sqrt{n}\cdot H_x\right)\left(\mathit{\Delta}_{x,j}^{-1}\left(v\right)\right)}{\varsigma_{dx}\left(\mathit{\Delta}_{x,j}^{-1}\left(v\right)\right)}\right|\right\}\leq\kappa,$$

wpa1. The desired property follows from this result, (A13) and (A14) (with h replaced by $h+\sqrt{n}\cdot H_x$), and the fact that $\sup_{u\in\mathscr{U}_n(v)}\rho_{d'x,j}\left(n^{-1/2}u+v\right)$ is bounded uniformly in $v\in\left[\underline{v}_x,\overline{v}_x\right]$, when n is sufficiently large.

By the property shown in the preceding paragraph, we can find some $\delta_0>0$ such that $\mathit{\Gamma}_n\left(\delta_0\right)\leq\varepsilon/2$ wpa1. Then, by this result, (B5) and (B6), we have

$$\begin{aligned}
&\Pr{}_{\dagger}\left[\sup_{v\in\left[\underline{v}_x,\overline{v}_x\right]}\left|\left(\mathbb{G}_n^{\dagger}+\mathbb{G}_n\right)\gamma_{dx}\left(\cdot\mid v,\sqrt{n}\cdot H_x^{\dagger},0\right)\right|>\varepsilon\right]\\
\leq\;&\Pr{}_{\dagger}\left[\sup_{h\in\bigcup_{k=1}^{p_{\delta_0}}B(h_k,\delta_0)}\sup_{v\in\left[\underline{v}_x,\overline{v}_x\right]}\left|\left(\mathbb{G}_n^{\dagger}+\mathbb{G}_n\right)\gamma_{dx}\left(\cdot\mid v,h+\sqrt{n}\cdot H_x,0\right)\right|>\varepsilon\right]\\
&+\Pr{}_{\dagger}\left[\sqrt{n}\left(H_x^{\dagger}-H_x\right)\notin\bigcup_{k=1}^{p_{\delta_0}}B\left(h_k,\delta_0\right)\right]\\
\leq\;&\Pr{}_{\dagger}\left[T_n^{\dagger}\left(\delta_0\right)+\mathit{\Gamma}_n\left(\delta_0\right)>\varepsilon\right]+\frac{\varepsilon}{2}\\
=\;&\Pr{}_{\dagger}\left[T_n^{\dagger}\left(\delta_0\right)+\mathit{\Gamma}_n\left(\delta_0\right)>\varepsilon\right]\mathbb{1}\left(\mathit{\Gamma}_n\left(\delta_0\right)>\frac{\varepsilon}{2}\right)+\Pr{}_{\dagger}\left[T_n^{\dagger}\left(\delta_0\right)+\mathit{\Gamma}_n\left(\delta_0\right)>\varepsilon\right]\mathbb{1}\left(\mathit{\Gamma}_n\left(\delta_0\right)\leq\frac{\varepsilon}{2}\right)+\frac{\varepsilon}{2}\\
\leq\;&\Pr{}_{\dagger}\left[T_n^{\dagger}\left(\delta_0\right)>\frac{\varepsilon}{2}\right]+\frac{\varepsilon}{2},
\end{aligned}$$

where the second and the last inequality hold wpa1, since $\mathbb{1}\left(\Gamma_n\left(\delta_0\right)>\varepsilon/2\right)=0$ wpa1. (B2) follows from this result and the fact that $T_n^{\dagger}\left(\delta_0\right)=o_p^{\dagger}\left(1\right)$, which implies that $\Pr_{\dagger}\left[T_n^{\dagger}\left(\delta\right)>\varepsilon/2\right]\leq\varepsilon/2$ occurs wpa1. ∎

Let $\widehat{\phi}_{dx}^{\dagger}\left(y\right)$ denote the leave-in version of $\widehat{\phi}_{dx}^{(-i)}\left(y\right)$. Under the assumptions in the main text, MMY show the following bootstrap analogue of the Bahadur-type representation result (A24) (see the proof of Lemma S6 of MMY):

$$\widehat{\phi}_{dx}^{\dagger}\left(y\right)-\phi_{dx}\left(y\right)=\frac{1}{n}\sum_{i=1}^{n}L_{dx}\left(W_i^{\dagger},y\right)+\xi_{dx}^{\dagger}\left(y\right),$$

where the remainder term on the right hand side satisfies the condition that $\Pr_{\dagger}\left[\left\|\xi_{dx}^{\dagger}\right\|_{\infty}>r_n\right]\leq r_n'$ wpa1 for some positive deterministic sequences $\{r_n\}_{n=1}^{\infty}$ and $\{r_n'\}_{n=1}^{\infty}$ that are proportional to $\left(\log\left(n\right)/n\right)^{3/4}$ and n^{-1} respectively. Then we have the following bootstrap analogue of (A25):

$$\widehat{\Delta}_i^{\dagger}-\Delta_i^{\dagger}=\left\{\frac{D_i^{\dagger}}{\varsigma_{0x}\left(\epsilon_i^{\dagger}\right)}+\frac{\left(1-D_i^{\dagger}\right)}{\varsigma_{1x}\left(\epsilon_i^{\dagger}\right)}\right\}H_x\left(\epsilon_i^{\dagger}\right)+\xi_i^{\dagger},$$

where $\bar{\xi}_n^{\dagger}:=\max\left\{\left|\xi_1^{\dagger}\right|,...,\left|\xi_n^{\dagger}\right|\right\}$ satisfies the condition that $\Pr_{\dagger}\left[\bar{\xi}_n^{\dagger}>r_n\right]=o_p\left(1\right)$ for some r_n proportional to $\left(\log\left(n\right)/n\right)^{3/4}$. Using these results and notations, we can show the following bootstrap analogue of Lemma 3.

Lemma 5. *Under the assumptions in the statement of Theorem 2,* $\widehat{F}_{\Delta|X}^{\dagger}\left(v\mid x\right)-\breve{F}_{\Delta|X}^{\dagger}\left(v\mid x\right)=o_p^{\dagger}\left(n^{-1/2}\right)$*, uniformly in* $v\in\left[\underline{v}_x,\overline{v}_x\right]$.

Proof of Lemma 5. By (A6) and the triangle inequality,

$$\begin{aligned}&\left|\sqrt{n}\left(\widehat{F}_{\Delta|X}^{\dagger}\left(v\mid x\right)-\breve{F}_{\Delta|X}^{\dagger}\left(v\mid x\right)\right)\right|\\&\quad\leq\sqrt{n}\left(\breve{F}_{\Delta|X}^{\dagger}\left(v+\bar{\xi}_n^{\dagger}\mid x\right)-\breve{F}_{\Delta|X}^{\dagger}\left(v\mid x\right)\right)+\sqrt{n}\left(\breve{F}_{\Delta|X}^{\dagger}\left(v\mid x\right)-\breve{F}_{\Delta|X}^{\dagger}\left(v-\bar{\xi}_n^{\dagger}\mid x\right)\right),\end{aligned}\tag{B9}$$

where $\bar{\xi}_n^{\dagger}$ satisfies $\Pr_{\dagger}\left[\bar{\xi}_n^{\dagger}>r_n\right]=o_p\left(1\right)$ for some $r_n=O\left(\left(\log\left(n\right)/n\right)^{3/4}\right)$.

By Lemma 2, the bootstrap functional delta method (Kosorok, 2007, Theorem 12.1 and Equation (12.1)) and (B1),

$$\sqrt{n}\left\{\Psi_{dx}\left(\Delta_x+\frac{H_x^{\dagger}}{\varsigma_{dx}}\right)-\Psi_{dx}\left(\Delta_x+\frac{H_x}{\varsigma_{dx}}\right)\right\}\left(v\right)=\psi_{dx}\left(\frac{\sqrt{n}\left(H_x^{\dagger}-H_x\right)}{\varsigma_{dx}}\right)\left(v\right)+o_p^{\dagger}\left(1\right),\tag{B10}$$

uniformly in $v\in\left[\underline{v}_x,\overline{v}_x\right]$. Then, by Lemma 4,

$$\breve{F}_{\Delta|X}^{\dagger}\left(v+r_n\mid x\right)-\breve{F}_{\Delta|X}^{\dagger}\left(v\mid x\right)$$

$$
\begin{aligned}
&= \sum_{d\in\{0,1\}} \left\{ \Psi_{dx}\left(\Delta_x + \frac{H_x^\dagger}{\varsigma_{dx}} - r_n\right) - \Psi_{dx}\left(\Delta_x + \frac{H_x^\dagger}{\varsigma_{dx}}\right)\right\}(v) + o_p^\dagger\left(n^{-1/2}\right) \\
&= \sum_{d\in\{0,1\}} \left\{ \Psi_{dx}\left(\Delta_x + \frac{H_x^\dagger}{\varsigma_{dx}} - r_n\right) - \Psi_{dx}\left(\Delta_x + \frac{H_x}{\varsigma_{dx}} - r_n\right)\right\}(v) \\
&\quad - \sum_{d\in\{0,1\}} \left\{ \Psi_{dx}\left(\Delta_x + \frac{H_x^\dagger}{\varsigma_{dx}}\right) - \Psi_{dx}\left(\Delta_x + \frac{H_x}{\varsigma_{dx}}\right)\right\}(v) \\
&\quad + \sum_{d\in\{0,1\}} \left\{ \Psi_{dx}\left(\Delta_x + \frac{H_x}{\varsigma_{dx}} - r_n\right) - \Psi_{dx}\left(\Delta_x + \frac{H_x}{\varsigma_{dx}}\right)\right\}(v) + o_p^\dagger\left(n^{-1/2}\right), \qquad \text{(B11)}
\end{aligned}
$$

uniformly in $v \in [\underline{v}_x, \overline{v}_x]$. It was shown in the proof of Lemma 3 that the third term on the right hand side of the second equality in (B11) is $o_p\left(n^{-1/2}\right)$. It follows from Slutsky's theorem, the bootstrap functional delta method and (B1) that the first term on the right hand side of the second equality in (B11) has the same linearization as the right hand side of (B10). It follows from these results that $\breve{F}^\dagger_{\Delta|X}(v + r_n \mid x) - \breve{F}^\dagger_{\Delta|X}(v \mid x) = o_p^\dagger\left(n^{-1/2}\right)$, uniformly in $v \in [\underline{v}_x, \overline{v}_x]$. By this result and the union bound, for any $\varepsilon > 0$,

$$
\begin{aligned}
& \Pr{}_\dagger\left[\sup_{v\in[\underline{v}_x,\overline{v}_x]} \sqrt{n}\left(\breve{F}^\dagger_{\Delta|X}\left(v + \bar{\xi}_n^\dagger \mid x\right) - \breve{F}^\dagger_{\Delta|X}(v \mid x)\right) > \varepsilon\right] \\
\leq\ & \Pr{}_\dagger\left[\sup_{v\in[\underline{v}_x,\overline{v}_x]} \sqrt{n}\left(\breve{F}^\dagger_{\Delta|X}(v + r_n \mid x) - \breve{F}^\dagger_{\Delta|X}(v \mid x)\right) > \varepsilon\right] + \Pr{}_\dagger\left[\bar{\xi}_n^\dagger > r_n\right] \\
=\ & o_p(1).
\end{aligned}
$$

By similar arguments, the second term on the right hand side of (B9) is also $o_p^\dagger(1)$. Thus, the conclusion of the lemma follows immediately from these results. ■

By using Lemmas 4 and 5, we now prove Theorem 2.

Proof of Theorem 2. First note that we have

$$
\begin{aligned}
S_F^\dagger(v \mid x) &= \sqrt{n}\left(\widehat{F}^\dagger_{\Delta|X}(v \mid x) - \widetilde{F}^\dagger_{\Delta|X}(v \mid x)\right) + \sqrt{n}\left(\widetilde{F}^\dagger_{\Delta|X}(v \mid x) - \widetilde{F}_{\Delta|X}(v \mid x)\right) \\
&\quad - \sqrt{n}\left(\widehat{F}_{\Delta|X}(v \mid x) - \widetilde{F}_{\Delta|X}(v \mid x)\right) \\
&= \sqrt{n}\sum_{d\in\{0,1\}} \left\{ \Psi_{dx}\left(\Delta_x + \frac{H_x^\dagger}{\varsigma_{dx}}\right) - \Psi_{dx}\left(\Delta_x + \frac{H_x}{\varsigma_{dx}}\right)\right\}(v) \\
&\quad + \sqrt{n}\left(\widetilde{F}^\dagger_{\Delta|X}(v \mid x) - \widetilde{F}_{\Delta|X}(v \mid x)\right) + o_p^\dagger(1) \\
&= \bar{S}_F^\dagger(v \mid x) + o_p^\dagger(1), \qquad \text{(B12)}
\end{aligned}
$$

uniformly in $v \in [\underline{v}_x, \overline{v}_x]$, where

$$\bar{S}_F^{\dagger}(v \mid x) := \sum_{d \in \{0,1\}} \psi_{dx} \left(\frac{\sqrt{n}\left(H_x^{\dagger} - H_x\right)}{\varsigma_{dx}} \right)(v) + \sqrt{n}\left(\widetilde{F}_{\Delta|X}^{\dagger}(v \mid x) - \widetilde{F}_{\Delta|X}(v \mid x)\right),$$

the second equality follows from Lemmas 1, 3, 4, 5 and (B1), the third equality follows from (B10). Note that we can write $\bar{S}_F^{\dagger}(v \mid x) = \mathbb{G}_n^{\dagger} \iota_x(\cdot \mid v)$. By Van der Vaart (2000, Theorem 23.7), we have $\bar{S}_F^{\dagger}(\cdot \mid x) \rightsquigarrow_{\dagger} \mathbb{F}(\cdot \mid x)$ in $\ell^{\infty}[\underline{v}_x, \overline{v}_x]$.

Since (B12) implies that for all $\varepsilon > 0$,

$$\beta(\varepsilon) := \Pr\left[\Pr_{\dagger}\left[\left\| S_F^{\dagger}(\cdot \mid x) - \bar{S}_F^{\dagger}(\cdot \mid x)\right\|_{\infty} > \varepsilon\right] > \varepsilon\right] \downarrow 0,$$

as $n \uparrow \infty$. By Pollard (2002, Lemma 22), there exists a deterministic sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ that converges to zero and also satisfies $\beta(\varepsilon_n) \downarrow 0$ as $n \uparrow \infty$. Therefore,

$$\Pr_{\dagger}\left[\left\| S_F^{\dagger}(\cdot \mid x) - \bar{S}_F^{\dagger}(\cdot \mid x)\right\|_{\infty} > \varepsilon_n\right] \leq \varepsilon_n, \tag{B13}$$

occurs wpa1. Then we have

$$\begin{aligned}
& \sup_{h \in BL_1} \left| \mathrm{E}_{\dagger}\left[h\left(S_F^{\dagger}(\cdot \mid x)\right)\right] - \mathrm{E}_{\dagger}\left[h\left(\bar{S}_F^{\dagger}(\cdot \mid x)\right)\right]\right| \\
\leq & \sup_{h \in BL_1} \left| \mathrm{E}_{\dagger}\left[\left(h\left(S_F^{\dagger}(\cdot \mid x)\right) - h\left(\bar{S}_F^{\dagger}(\cdot \mid x)\right)\right) \mathbb{1}\left(\left\| S_F^{\dagger}(\cdot \mid x) - \bar{S}_F^{\dagger}(\cdot \mid x)\right\|_{\infty} > \varepsilon_n\right)\right]\right| \\
& + \sup_{h \in BL_1} \left| \mathrm{E}_{\dagger}\left[\left(h\left(S_F^{\dagger}(\cdot \mid x)\right) - h\left(\bar{S}_F^{\dagger}(\cdot \mid x)\right)\right) \mathbb{1}\left(\left\| S_F^{\dagger}(\cdot \mid x) - \bar{S}_F^{\dagger}(\cdot \mid x)\right\|_{\infty} \leq \varepsilon_n\right)\right]\right| \\
\leq & \; 3 \cdot \varepsilon_n,
\end{aligned} \tag{B14}$$

where the first inequality follows from the triangle inequality and the second inequality holds wpa1 and follows from the the definition of BL_1, which is the shorthand notation for$BL_1(\ell^{\infty}[\underline{v}_x, \overline{v}_x])$ here, and (B13). By $\bar{S}_F^{\dagger}(\cdot \mid x) \rightsquigarrow_{\dagger} \mathbb{F}(\cdot \mid x)$ and using (B14), we have $S_F^{\dagger}(\cdot \mid x) \rightsquigarrow_{\dagger} \mathbb{F}(\cdot \mid x)$ in $\ell^{\infty}[\underline{v}_x, \overline{v}_x]$. ∎

Proof of Corollary 2. By Theorem 1(ii) and Van der Vaart (2000, Lemma 2.11), we have

$$\sup_{u \in \mathbb{R}} \left| \Pr\left[S_F(v \mid x) \leq u\right] - \Pr\left[\mathbb{F}(v \mid x) \leq u\right]\right| \to 0, \tag{B15}$$

as $n \uparrow \infty$, and since the CDF of $\|\mathbb{F}(\cdot \mid x)\|_{\infty}$ is continuous everywhere on $\mathbb{R}$, by Theorem 1(i) and CMT, we also have

$$\sup_{u \in \mathbb{R}} \left| \Pr\left[\|S_F(\cdot \mid x)\|_{\infty} \leq u\right] - \Pr\left[\|\mathbb{F}(\cdot \mid x)\|_{\infty} \leq u\right]\right| \to 0, \tag{B16}$$

as $n \uparrow \infty$.

For $p \in (0,1)$, let

$$\tilde{s}_{F,p}(v \mid x) := \inf\left\{u \in \mathbb{R} : \Pr_{\dagger}\left[S_F^{\dagger}(v \mid x) \leq u\right] \geq p\right\} \tag{B17}$$

be the p-th quantile of the resampling distribution of $S_F^{\dagger}(v \mid x)$. Since quantiles are equivariant to monotone transformations, we have

$$\begin{aligned} s_{F,\alpha/2}(v \mid x) &= \widehat{F}_{\Delta|X}(v \mid x) + \frac{\tilde{s}_{F,\alpha/2}(v \mid x)}{\sqrt{n}} \\ s_{F,1-\alpha/2}(v \mid x) &= \widehat{F}_{\Delta|X}(v \mid x) + \frac{\tilde{s}_{F,1-\alpha/2}(v \mid x)}{\sqrt{n}}. \end{aligned}$$

Then,

$$\begin{aligned} \Pr\left[F_{\Delta|X}(v \mid x) \in \left[s_{F,\alpha/2}(v \mid x), s_{F,1-\alpha/2}(v \mid x)\right]\right] &= \Pr\left[-S_F(v \mid x) \leq \tilde{s}_{F,1-\alpha/2}(v \mid x)\right] \\ &\quad - \left\{1 - \Pr\left[S_F(v \mid x) \leq -\tilde{s}_{F,\alpha/2}(v \mid x)\right]\right\}. \end{aligned} \tag{B18}$$

Let $q(p)$ denote the p-th quantile of $\mathbb{F}(v \mid x)$ for any $p \in (0,1)$. By (31) and Pollard (2002, Lemma 22), there exists some $\varepsilon_n \downarrow 0$ such that for all $p \in (0,1)$,

$$\begin{aligned} o(1) &= \Pr\left[\sup_{u \in \mathbb{R}}\left|\Pr_{\dagger}\left[S_F^{\dagger}(v \mid x) \leq u\right] - \Pr\left[\mathbb{F}(v \mid x) \leq u\right]\right| > \varepsilon_n\right] \\ &\geq \Pr\left[\left|\Pr_{\dagger}\left[S_F^{\dagger}(v \mid x) \leq q(p+\varepsilon_n)\right] - \Pr\left[\mathbb{F}(v \mid x) \leq q(p+\varepsilon_n)\right]\right| > \varepsilon_n\right] \\ &\geq \Pr\left[\Pr_{\dagger}\left[S_F^{\dagger}(v \mid x) \leq q(p+\varepsilon_n)\right] < p\right] \\ &\geq \Pr\left[\tilde{s}_{F,p}(v \mid x) > q(p+\varepsilon_n)\right], \end{aligned} \tag{B19}$$

where the second inequality follows from Van der Vaart (2000, Lemma 21.1(ii)), and the third inequality follows from the definition (B17) of $\tilde{s}_{F,p}(v \mid x)$ and Van der Vaart (2000, Lemma 21.1(i)). By similar arguments, there exists some $\varepsilon_n' \downarrow 0$ such that for all $p \in (0,1)$,

$$\begin{aligned} o(1) &= \Pr\left[\left|\Pr_{\dagger}\left[S_F^{\dagger}(v \mid x) \leq q\left(p-\varepsilon_n'\right)\right] - \Pr\left[\mathbb{F}(v \mid x) \leq q\left(p-\varepsilon_n'\right)\right]\right| \geq \varepsilon_n'\right] \\ &\geq \Pr\left[\Pr_{\dagger}\left[S_F^{\dagger}(v \mid x) \leq q\left(p-\varepsilon_n'\right)\right] \geq p\right] \\ &\geq \Pr\left[\tilde{s}_{F,p}(v \mid x) \leq q\left(p-\varepsilon_n'\right)\right]. \end{aligned} \tag{B20}$$

It is clear that since $\mathbb{F}(v \mid x)$ has a normal distribution, q is continuous everywhere on $(0,1)$. By this result, (B19) and (B20), $\tilde{s}_{F,p}(v \mid x) \to_p q(p)$ for all $p \in (0,1)$. By this result, Slutsky's theorem and the symmetry of the distribution of $\mathbb{F}(v \mid x)$, we have $\Pr\left[-S_F(v \mid x) \leq \tilde{s}_{F,1-\alpha/2}(v \mid x)\right] \to 1-\alpha/2$ and $\Pr\left[S_F(v \mid x) \leq -\tilde{s}_{F,\alpha/2}(v \mid x)\right] \to 1-\alpha/2$. The conclusion in Part (i) follows from these facts and (B18).

Now we prove Part (ii). It is easy to see that

$$\Pr\left[F_{\Delta|X}(v \mid x) \in CB_F(v \mid x),\ \forall v \in [\underline{v}_x, \overline{v}_x]\right] = \Pr\left[\|S_F(\cdot \mid x)\|_\infty \le s^{\mathsf{unif}}_{F,1-\alpha}\right]. \tag{B21}$$

Let $q^{\mathsf{unif}}(p)$ denote the p-th quantile of $\|\mathbb{F}(\cdot \mid x)\|_\infty$ for any $p \in (0,1)$. By (32), the continuity of the CDF of $\|\mathbb{F}(\cdot \mid x)\|_\infty$ and the same arguments used to show (B19) and (B20), for any $p \in (0,1)$, there exists some $\varepsilon_n, \varepsilon_n' \downarrow 0$ such that $\Pr\left[s^{\mathsf{unif}}_{F,p} > q^{\mathsf{unif}}(p+\varepsilon_n)\right] = o(1)$ and $\Pr\left[s^{\mathsf{unif}}_{F,p} \le q^{\mathsf{unif}}(p-\varepsilon_n')\right] = o(1)$. By these results, we have

$$\begin{aligned}
\Pr\left[\|S_F(\cdot \mid x)\|_\infty > s^{\mathsf{unif}}_{F,1-\alpha}\right] &= \Pr\left[\|S_F(\cdot \mid x)\|_\infty > s^{\mathsf{unif}}_{F,1-\alpha}, s^{\mathsf{unif}}_{F,1-\alpha} \le q^{\mathsf{unif}}\left(1-\alpha-\varepsilon_n'\right)\right] \\
&\quad + \Pr\left[\|S_F(\cdot \mid x)\|_\infty > s^{\mathsf{unif}}_{F,1-\alpha}, s^{\mathsf{unif}}_{F,1-\alpha} > q^{\mathsf{unif}}\left(1-\alpha-\varepsilon_n'\right)\right] \\
&\le \Pr\left[\|S_F(\cdot \mid x)\|_\infty > q^{\mathsf{unif}}\left(1-\alpha-\varepsilon_n'\right)\right] + o(1) \\
&= \Pr\left[\|\mathbb{F}(\cdot \mid x)\|_\infty > q^{\mathsf{unif}}\left(1-\alpha-\varepsilon_n'\right)\right] + o(1) \\
&= \alpha + o(1),
\end{aligned} \tag{B22}$$

where the second equality follows from (B16) and the third equality follows from Van der Vaart (2000, Lemma 21.1(ii)) and the continuity of the CDF of $\|\mathbb{F}(\cdot \mid x)\|_\infty$. By using the same arguments, we can show that

$$\begin{aligned}
\Pr\left[\|S_F(\cdot \mid x)\|_\infty \le s^{\mathsf{unif}}_{F,1-\alpha}\right] &= \Pr\left[\|S_F(\cdot \mid x)\|_\infty \le s^{\mathsf{unif}}_{F,1-\alpha}, s^{\mathsf{unif}}_{F,1-\alpha} > q^{\mathsf{unif}}(1-\alpha+\varepsilon_n)\right] \\
&\quad + \Pr\left[\|S_F(\cdot \mid x)\|_\infty \le s^{\mathsf{unif}}_{F,1-\alpha}, s^{\mathsf{unif}}_{F,1-\alpha} \le q^{\mathsf{unif}}(1-\alpha+\varepsilon_n)\right] \\
&\le \Pr\left[\|S_F(\cdot \mid x)\|_\infty \le q^{\mathsf{unif}}(1-\alpha+\varepsilon_n)\right] + o(1) \\
&= \Pr\left[\|\mathbb{F}(\cdot \mid x)\|_\infty \le q^{\mathsf{unif}}(1-\alpha+\varepsilon_n)\right] + o(1) \\
&= 1-\alpha + o(1).
\end{aligned}$$

It follows from this result and (B22) that $\Pr\left[\|S_F(\cdot \mid x)\|_\infty \le s^{\mathsf{unif}}_{F,1-\alpha}\right] \to 1-\alpha$ as $n \uparrow \infty$. The conclusion in Part (ii) follows from this result and (B21). ∎

Proof of Corollary 3. By using Theorem 2(i) and the bootstrap analogue of the CMT, we have $S^{\dagger}_F(v \mid x) \rightsquigarrow_{\dagger} \mathbb{F}(v \mid x)$. By this result and similar arguments, we can easily show that a result similar to (B4) also holds for $S^{\dagger}_F(v \mid x)$. It follows from this result and the fact that $\mathbb{F}(v \mid x)$ is a normal random variable that for all $\varepsilon > 0$, we can find some $M > 0$ such that $\Pr_{\dagger}\left[\left|S^{\dagger}_F(v \mid x)\right| \ge M\right] < \varepsilon$ wpa1. It easily follows that $\widehat{F}^{\dagger}_{\Delta|X}(v \mid x) - \widehat{F}_{\Delta|X}(v \mid x) = o^{\dagger}_p(1)$ for all v in the interior of $\mathscr{S}_{\Delta|X=x}$. It follows from this result, $\widehat{F}_{\Delta|X}(v \mid x) - F_{\Delta|X}(v \mid x) = o_p(1)$ and (B1) that $\widehat{F}^{\dagger}_{\Delta|X}(v \mid x) - F_{\Delta|X}(v \mid x) = o^{\dagger}_p(1)$. By this result and an inequality similar to (A33), we have $\Pr_{\dagger}\left[\widehat{Q}^{\dagger}_{\Delta|X}(\tau \mid x) \le Q_{\Delta|X}(\tau \mid x) + \varepsilon\right] \to_p 1$ as $n \uparrow \infty$ for all sufficiently small $\varepsilon > 0$ such that $Q_{\Delta|X}(\tau \mid x) + \varepsilon$ is an interior point of $\mathscr{S}_{\Delta|X=x}$. By similar arguments, for all sufficiently small $\varepsilon > 0$ such that $Q_{\Delta|X}(\tau \mid x) - \varepsilon$ is an interior point of $\mathscr{S}_{\Delta|X=x}$, $\Pr_{\dagger}\left[\widehat{Q}^{\dagger}_{\Delta|X}(\tau \mid x) \geq Q_{\Delta|X}(\tau \mid x) - \varepsilon\right] \rightarrow_p 1$ as $n \uparrow \infty$.

Let $[a,b]$ be the interval defined in the proof of Corollary 1. Then, it follows that

$$\Pr_{\dagger}\left[\widehat{Q}^{\dagger}_{\Delta|X}(\overline{\tau} \mid x) < b \text{ and } \widehat{Q}^{\dagger}_{\Delta|X}(\underline{\tau} \mid x) > a\right] \rightarrow_p 1. \tag{B23}$$

Similarly, if $\widehat{Q}^{\dagger}_{\Delta|X}(\overline{\tau} \mid x) < b$ and $\widehat{Q}^{\dagger}_{\Delta|X}(\underline{\tau} \mid x) > a$, for all $\tau \in [\underline{\tau}, \overline{\tau}]$, $\widehat{Q}^{\dagger}_{\Delta|X}(\tau \mid x)$ can be written as $\phi_\tau\left(\widehat{F}^{\dagger}_{\Delta|X}(\cdot \mid x)\Big|_{[a,b]}\right)$. Let $\dot{F}^{\dagger}_{\Delta|X}(v \mid x)$ be $\widehat{F}^{\dagger}_{\Delta|X}(v \mid x)$ for all $v \in \mathbb{R}$ if $\widehat{Q}^{\dagger}_{\Delta|X}(\underline{\tau} \mid x) > a$ and $\widehat{Q}^{\dagger}_{\Delta|X}(\overline{\tau} \mid x) < b$ and let $\dot{F}^{\dagger}_{\Delta|X}(v \mid x)$ be $F_{\Delta|X}(v \mid x)$ for all $v \in \mathbb{R}$ otherwise. Then, since for any $\varepsilon > 0$ and $\alpha_n \downarrow 0$,

$$\Pr_{\dagger}\left[\sup_{\tau \in [\underline{\tau}, \overline{\tau}]}\left|\widehat{Q}^{\dagger}_{\Delta|X}(\tau \mid x) - \phi_\tau\left(\dot{F}^{\dagger}_{\Delta|X}(\cdot \mid x)\Big|_{[a,b]}\right)\right| \leq \varepsilon\alpha_n\right] \geq \Pr_{\dagger}\left[\widehat{Q}^{\dagger}_{\Delta|X}(\overline{\tau} \mid x) < b \text{ and } \widehat{Q}^{\dagger}_{\Delta|X}(\underline{\tau} \mid x) > a\right],$$

by (B23), we have

$$\widehat{Q}^{\dagger}_{\Delta|X}(\tau \mid x) - \phi_\tau\left(\dot{F}^{\dagger}_{\Delta|X}(\cdot \mid x)\Big|_{[a,b]}\right) = o^{\dagger}_p(\alpha_n), \tag{B24}$$

uniformly in $\tau \in [\underline{\tau}, \overline{\tau}]$, for any $\alpha_n \downarrow 0$. And, similarly, $\widehat{F}^{\dagger}_{\Delta|X}(v \mid x) - \dot{F}^{\dagger}_{\Delta|X}(v \mid x) = o^{\dagger}_p(\alpha_n)$, uniformly in $v \in [a,b]$, for all $\alpha_n \downarrow 0$. By this result, (A35) and (B1),

$$\sqrt{n}\left(\dot{F}^{\dagger}_{\Delta|X}(v \mid x) - \dot{F}_{\Delta|X}(v \mid x)\right) = S^{\dagger}_F(v \mid x) + o^{\dagger}_p(1), \tag{B25}$$

uniformly in $v \in [a,b]$. By this result, Theorem 2 with $[\underline{v}_x, \overline{v}_x] = [a,b]$ and also the the same argument as that in the proof of Theorem 2, $\sqrt{n}\left(\dot{F}^{\dagger}_{\Delta|X}(\cdot \mid x)\Big|_{[a,b]} - \dot{F}_{\Delta|X}(\cdot \mid x)\Big|_{[a,b]}\right)$ converges in distribution in $\ell^\infty[a,b]$ to the same limit as that of $\sqrt{n}\left(\dot{F}_{\Delta|X}(\cdot \mid x)\Big|_{[a,b]} - F_{\Delta|X}(\cdot \mid x)\Big|_{[a,b]}\right)$. A bootstrap analogue of (A36), i.e.,

$$\begin{aligned} S^{\dagger}_Q(\tau \mid x) &= \sqrt{n}\left\{\phi_\tau\left(\dot{F}^{\dagger}_{\Delta|X}(\cdot \mid x)\Big|_{[a,b]}\right) - \phi_\tau\left(\dot{F}_{\Delta|X}(\cdot \mid x)\Big|_{[a,b]}\right)\right\} + o^{\dagger}_p(1) \\ &= -\frac{S^{\dagger}_F\left(Q_{\Delta|X}(\tau \mid x) \mid x\right)}{f_{\Delta|X}\left(Q_{\Delta|X}(\tau \mid x) \mid x\right)} + o^{\dagger}_p(1), \end{aligned}$$

uniformly in $\tau \in [\underline{\tau}, \overline{\tau}]$, follows from this result, (A34), (B24), Van der Vaart (2000, Lemma 21.4), the bootstrap version of the functional delta method (Kosorok, 2007, Theorem 12.1 and Equation (12.1)), (B25) and (B1). The conclusion in Part (i) follows easily from the CMT. Conclusions in Parts (ii) and (iii) follow from the same arguments as those used in the proof of Corollary 2. ∎

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