November 27, 2018

Restricted LS

Consider the *restricted* LS problem

$$\min_{\boldsymbol{b}} \left(\boldsymbol{Y}-\boldsymbol{X}\boldsymbol{b}\right)'\left(\boldsymbol{Y}-\boldsymbol{X}\boldsymbol{b}\right) \quad \text{s.t. } \boldsymbol{R}\boldsymbol{b}=\boldsymbol{r}.$$

A Lagrangian function for this problem is

$$L(\boldsymbol{b},\boldsymbol{\lambda}) = \left(\boldsymbol{Y}-\boldsymbol{X}\boldsymbol{b}\right)'\left(\boldsymbol{Y}-\boldsymbol{X}\boldsymbol{b}\right) + 2\boldsymbol{\lambda}'\left(\boldsymbol{R}\boldsymbol{b}-\boldsymbol{r}\right),$$

where $\boldsymbol{\lambda}$ is a q-vector. Let $\widetilde{\boldsymbol{\beta}}, \widetilde{\boldsymbol{\lambda}}$ be the solution, where $\widetilde{\boldsymbol{\beta}}$ is the restricted LS estimator. It has to satisfy the first-order conditions

$$\frac{\partial L\left(\widetilde{\boldsymbol{\beta}},\widetilde{\boldsymbol{\lambda}}\right)}{\partial \boldsymbol{b}} = 2\boldsymbol{X}'\boldsymbol{X}\widetilde{\boldsymbol{\beta}} - 2\boldsymbol{X}'\boldsymbol{Y} + 2\boldsymbol{R}'\widetilde{\boldsymbol{\lambda}} = \boldsymbol{0}, \tag{1}$$

$$\frac{\partial L\left(\widetilde{\boldsymbol{\beta}},\widetilde{\boldsymbol{\lambda}}\right)}{\partial \boldsymbol{\lambda}} = \boldsymbol{R}\widetilde{\boldsymbol{\beta}} - \boldsymbol{r} = \boldsymbol{0}. \tag{2}$$

From (1),

$$\begin{aligned}
\widetilde{\boldsymbol{\beta}} &= \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}\left(\boldsymbol{X}'\boldsymbol{Y} - \boldsymbol{R}'\widetilde{\boldsymbol{\lambda}}\right) \\
&= \widehat{\boldsymbol{\beta}} - \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}\boldsymbol{R}'\widetilde{\boldsymbol{\lambda}}.
\end{aligned}$$

Combining the last equation with (2),

$$\begin{aligned}
\boldsymbol{r} &= \boldsymbol{R}\widetilde{\boldsymbol{\beta}} \\
&= \boldsymbol{R}\widehat{\boldsymbol{\beta}} - \boldsymbol{R}\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}\boldsymbol{R}'\widetilde{\boldsymbol{\lambda}},
\end{aligned}$$

and

$$\widetilde{\boldsymbol{\lambda}} = \left(\boldsymbol{R}\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}\boldsymbol{R}'\right)^{-1}\left(\boldsymbol{R}\widehat{\boldsymbol{\beta}} - r\right).$$

Therefore, the restricted LS estimator is given by

$$\widetilde{\boldsymbol{\beta}} = \widehat{\boldsymbol{\beta}} - \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}\boldsymbol{R}'\left(\boldsymbol{R}\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}\boldsymbol{R}'\right)^{-1}\left(\boldsymbol{R}\widehat{\boldsymbol{\beta}} - r\right).$$

Define the restricted residuals

$$\begin{aligned}
\widetilde{\boldsymbol{e}} &= \boldsymbol{Y} - \boldsymbol{X}\widetilde{\boldsymbol{\beta}} \\
&= \left(\boldsymbol{Y} - \boldsymbol{X}\widehat{\boldsymbol{\beta}}\right) + \boldsymbol{X}\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}\boldsymbol{R}'\left(\boldsymbol{R}\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}\boldsymbol{R}'\right)^{-1}\left(\boldsymbol{R}\widehat{\boldsymbol{\beta}} - r\right) \\
&= \widehat{\boldsymbol{e}} + \boldsymbol{X}\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}\boldsymbol{R}'\left(\boldsymbol{R}\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}\boldsymbol{R}'\right)^{-1}\left(\boldsymbol{R}\widehat{\boldsymbol{\beta}} - r\right),
\end{aligned}$$

where $\widehat{\boldsymbol{e}} = \boldsymbol{Y} - \boldsymbol{X}\widehat{\boldsymbol{\beta}}$ is the vector of unrestricted residuals. Consider the *restricted* Residual Sum of Squares:

$$\begin{aligned} RSS_r &= \widetilde{\boldsymbol{e}}'\widetilde{\boldsymbol{e}} \\ &= \widehat{\boldsymbol{e}}'\widehat{\boldsymbol{e}} + \left(\boldsymbol{R}\widehat{\boldsymbol{\beta}} - \boldsymbol{r}\right)' \left(\boldsymbol{R}\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}\boldsymbol{R}'\right)^{-1} \left(\boldsymbol{R}\widehat{\boldsymbol{\beta}} - \boldsymbol{r}\right) \\ &\quad + 2\widehat{\boldsymbol{e}}'\boldsymbol{X}\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}\boldsymbol{R}'\left(\boldsymbol{R}\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}\boldsymbol{R}'\right)^{-1}\left(\boldsymbol{R}\widehat{\boldsymbol{\beta}} - \boldsymbol{r}\right) \\ &= RSS_{ur} + \left(\boldsymbol{R}\widehat{\boldsymbol{\beta}} - \boldsymbol{r}\right)' \left(\boldsymbol{R}\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}\boldsymbol{R}'\right)^{-1} \left(\boldsymbol{R}\widehat{\boldsymbol{\beta}} - \boldsymbol{r}\right), \end{aligned}$$

where $RSS_{ur} = \widehat{\boldsymbol{e}}'\widehat{\boldsymbol{e}}$ denotes the unrestricted Residual Sum of Squares. The F-statistic is

$$F = \frac{\left(RSS_r - RSS_{ur}\right)/q}{RSS_{ur}/(n-k)}. \tag{3}$$

Since $s^2 = \widehat{\boldsymbol{e}}'\widehat{\boldsymbol{e}}/(n-k) = RSS_{ur}/\left(n-k\right)$, the F-statistic can also be written as

$$F = \left(\boldsymbol{R}\widehat{\boldsymbol{\beta}} - \boldsymbol{r}\right)' \left(s^2\boldsymbol{R}\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}\boldsymbol{R}'\right)^{-1} \left(\boldsymbol{R}\widehat{\boldsymbol{\beta}} - \boldsymbol{r}\right)/q.$$