Introduction to Statistical Machine Learning with Applications in Econometrics

Lecture 3: Linear Regression (ISL ch. 3)

Instructor: Ma, Jun

Renmin University of China

September 23, 2021

Linear regression

- Linear regression is a simple approach to supervised learning. In particular, linear regression is a useful tool for predicting a quantitative response.
- The `Advertising` data has sales as the response (Y) and advertising budgets for TV (X_1), radio (X_2), and newspaper media (X_3) as predictors. A statistical model: $Y = f(X) + \epsilon$ with ϵ being independent of $X = (X_1, X_2, X_3)^\top$.
- Interesting questions:
  - Is there a relationship between advertising budget and sales? (Is $f(x_1, x_2, x_3) = \mathrm{E}[Y \mid X_1 = x_1, X_2 = x_2, X_3 = x_3]$ constant?)
  - How strong is the relationship between advertising budget and sales? (Variance of ϵ?)
  - Which media contribute to sales? (Partial derivatives of $f(x_1, x_2, x_3)$?)
  - How accurately can we predict future sales? (MSE of prediction for an unseen data point.)
  - Is the relationship ($f(x)$) linear?
  - Is there synergy (interaction) among the advertising media? ($\partial f(x_1, x_2, x_3) / \partial x_1$ depends on (x_2, x_3)?)

- linear regression model: $Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \epsilon$
  - ϵ is the error term that is independent of X.
  - β_0 and $(\beta_1, \beta_2, \beta_3)$ are intercept and slopes, which are also called coefficients.
- From the prediction perspective, essentially the model specifies a functional form for $f(X)$ and recovering f reduces to recovering the coefficients.
- From the causal inference perspective, essentially the model assumes that the effects are constant and there is no endogeneity issue.

Simple linear regression

- Simple linear regression model with a single predictor X: $Y = \beta_0 + \beta_1 X + \epsilon$.
- We have the training data:

$$(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n).$$

- Given some estimates $\hat{\beta}_0$ and $\hat{\beta}_1$ for the coefficients, for the unseen data point (X_0, Y_0), we predict Y_0 using $\hat{Y}_0 = \hat{\beta}_0 + \hat{\beta}_1 X_0$.
- Let $\hat{Y}_i = b_0 + b_1 X_i$ be the in-sample prediction for Y_i based on the i-th value of X_i.
- $e_i = Y_i - \hat{Y}_i$ represents the i-th residual and we the residual sum of squares (RSS) as

$$\begin{aligned}
\text{RSS} &= e_1^2 + e_2^2 + \cdots + e_n^2 \\
&= (Y_1 - b_0 - b_1 X_1)^2 + (Y_2 - b_0 - b_1 X_2)^2 + \\
&\quad \cdots + (Y_n - b_0 - b_1 X_n)^2 .
\end{aligned}$$

- The least squares approach chooses b_0 and b_1 to minimize the RSS. The minimizing values can be shown to be

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}\left(X_i - \overline{X}\right)\left(Y_i - \overline{Y}\right)}{\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2}$$

and $\hat{\beta}_0 = \overline{Y} - \hat{\beta}_1\overline{X}$, where $\overline{X} = n^{-1}\sum_{i=1}^{n} X_i$ and $\overline{Y} = n^{-1}\sum_{i=1}^{n} Y_i$.

Assessing the accuracy

- The standard error of an estimator reflects how it varies under repeated sampling:

$$\mathrm{SE}\left(\hat{\beta}_0\right)^2 = \sigma^2 \left[\frac{1}{n} + \frac{\overline{X}^2}{\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2}\right] \text{ and } \mathrm{SE}\left(\hat{\beta}_1\right)^2 = \frac{\sigma^2}{\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2},$$

  where $\sigma^2 = \mathrm{Var}\left[\epsilon\right]$.
- In general, σ^2 is not known, but can be estimated from the data.
- The estimate of σ ($\hat{\sigma}$) is known as the residual standard error:

$$\mathrm{RSE} = \sqrt{\frac{1}{n-2}\mathrm{RSS}} = \sqrt{\frac{1}{n-2}\sum_{i=1}^{n}\left(Y_i - \hat{Y}_i\right)^2},$$

  where the residual sum of squares: $\mathrm{RSS} = \sum_{i=1}^{n}\left(Y_i - \hat{Y}_i\right)^2$.

- Standard errors

$$\widehat{\mathrm{SE}}\left(\hat{\beta}_0\right)^2 = \hat{\sigma}^2 \left[\frac{1}{n} + \frac{\overline{X}^2}{\sum_{i=1}^n \left(X_i - \overline{X}\right)^2}\right] \text{ and } \widehat{\mathrm{SE}}\left(\hat{\beta}_1\right)^2 = \frac{\hat{\sigma}^2}{\sum_{i=1}^n \left(X_i - \overline{X}\right)^2}.$$

  can be used to compute confidence intervals.
- A 95% confidence interval is defined as an interval such that with 95% probability, the interval contains the true unknown value of the parameter.
- Approximately, with 95% probability

$$\left[\hat{\beta}_1 - 2 \cdot \widehat{\mathrm{SE}}\left(\hat{\beta}_1\right), \hat{\beta}_1 + 2 \cdot \widehat{\mathrm{SE}}\left(\hat{\beta}_1\right)\right]$$

  contains β_1, in a hypothetical scenario where we have repeated samples.

Hypothesis testing

- Standard errors can also be used to perform hypothesis tests on the coefficients. The most common hypothesis test involves testing the null hypothesis of

$$H_0\colon \text{There is no relationship between } X \text{ and } Y$$

against the alternative hypothesis

$$H_a\colon \text{There is some relationship between } X \text{ and } Y.$$

- This corresponds to testing $H_0 : \beta_1 = 0$ again $H_a : \beta_1 \neq 0$.
- We compute a t-statistic, given by

$$t = \frac{\hat{\beta}_1 - 0}{\widehat{\mathrm{SE}}\left(\hat{\beta}_1\right)},$$

which has a t-distribution with $n - 2$ degrees of freedom.
- p-value: the probability of observing any value equal to $|t|$ or larger.

Assessing the overall accuracy

- RSE is considered a measure of the lack of (in-sample) fit of the model to the data.
  - If the (in-sample) predictions $\hat{Y}_i$ are very close to the true outcome values Y_i, RSE will be small.
  - If $\hat{Y}_i$ is very far from Y_i for one or more observations, then the RSE may be quite large.
- R^2: the fraction of variance of Y explained by the model:

$$R^2 = \frac{\text{TSS} - \text{RSS}}{\text{TSS}} = 1 - \frac{\text{RSS}}{\text{TSS}},$$

  where $\text{TSS} = \sum_{i=1}^n \left(Y_i - \overline{Y}\right)^2$ is the total sum of squares.
- In simple linear regression, R^2 is the square of the sample correlation of X and Y:

$$R^2 = \left\{ \frac{\sum_{i=1}^n \left(X_i - \overline{X}\right)\left(Y_i - \overline{Y}\right)}{\sqrt{\sum_{i=1}^n \left(X_i - \overline{X}\right)^2}\sqrt{\sum_{i=1}^n \left(Y_i - \overline{Y}\right)^2}} \right\}^2 .$$

Multiple linear regression

- The multiple linear regression:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \cdots \beta_p X_p + \epsilon.$$

- We interpret β_j as the average effect on Y of a one unit increase in X_j, holding all other predictors fixed.
- Our training data:

$$\left\{ \left(Y_1, X_{1,1}, ..., X_{p,1}\right), \left(Y_2, X_{1,2}, ..., X_{p,2}\right), ..., \left(Y_n, X_{1,n}, ..., X_{p,n}\right) \right\}.$$

- Given estimates $b_0, b_1, ..., b_p$, we make in-sample predictions using:

$$\hat{Y}_i = b_0 + b_1 X_{1,i} + b_2 X_{2,i} + \cdots + b_p X_{p,i}.$$

- The values $\hat{\beta}_0, \hat{\beta}_1, ..., \hat{\beta}_p$ that minimize RSS are the multiple least squares regression coefficient estimates:

$$\begin{aligned} \text{RSS} &= \sum_{i=1}^{n} \left(Y_i - \hat{Y}_i\right)^2 \\ &= \sum_{i=1}^{n} \left(Y_i - b_0 - b_1 X_{1,i} - b_2 X_{2,i} - \cdots - b_p X_{p,i}\right)^2. \end{aligned}$$

	Coefficient	Std. Error	t-statistic	p-value
`Intercept`	12.351	0.621	19.88	< 0.0001
`newspaper`	0.055	0.017	3.30	0.00115

ISL Table 3.3: simple regression of `sales` on `newspaper`

	Coefficient	Std. Error	t-statistic	p-value
`Intercept`	2.939	0.3119	9.42	< 0.0001
`TV`	0.046	0.0014	32.81	< 0.0001
`radio`	0.189	0.0086	21.89	< 0.0001
`newspaper`	–0.001	0.0059	–0.18	0.8599

ISL Table 3.4: multiple regression

- The `newspaper` simple regression coefficient estimate was significantly non-zero, the multiple regression coefficient estimate for `newspaper` is close to zero, and the corresponding p-value is no longer significant.

	TV	radio	newspaper	sales
TV	1.0000	0.0548	0.0567	0.7822
radio		1.0000	0.3541	0.5762
newspaper			1.0000	0.2283
sales				1.0000

ISL Table 3.5

- The sample correlation between radio and newspaper is 0.35. Markets with high newspaper advertising tend to also have high radio advertising.
- Suppose that the multiple regression is correct and newspaper advertising is not associated with sales, but radio advertising is associated with sales.
- In a simple linear regression, we will observe that higher values of newspaper tend to be associated with higher values of sales, even though newspaper advertising is not directly associated with sales.

- Important questions:
  - Is at least one of the predictors $X_1, X_2, ..., X_p$ useful in predicting the response? (Model significance test.)
  - Do all the predictors help to explain Y, or is only a subset of the predictors useful? (Model selection will be discussed later in the class.)
  - How well does the model fit the data? (In-sample fit, measured by R^2.)
  - Given a set of predictor values, what response value should we predict, and how accurate is our prediction? (MSE of prediction for an unseen data point; is the linear model good enough for our prediction purpose?)

Model significance test

- Test that none of the regressors explain Y :

$$\begin{aligned} H_0 &: \quad \beta_1 = \beta_2 = \ldots = \beta_p = 0 \\ H_a &: \quad \text{at least one } \beta_j \text{ is non-zero.} \end{aligned}$$

- Use the F-statistic

$$F = \frac{(\mathrm{TSS} - \mathrm{RSS})\,/p}{\mathrm{RSS}/(n-p-1)} \sim F_{p,n-p-1}$$

under H_0. We expect F to be large if H_a is true.

Test of subset significance

- Sometimes we want to test that a particular subset of q of the coefficients are zero: $H_0 : \beta_{p-q+1} = \beta_{p-q+2} = \cdots \beta_p = 0$ against H_a: $\beta_{p-q+1} \neq 0$ or $\beta_{p-q+1} \neq 0$ or $\cdots$ or $\beta_p \neq 0$.
- We fit a second model that uses all the variables except those last q. Suppose that the residual sum of squares for that model is RSS_0.
- Use the F-statistic

$$F = \frac{(\text{RSS}_0 - \text{RSS})/q}{\text{RSS}/(n-p-1)} \sim F_{q,n-p-1}.$$

Qualitative predictors

- Some predictors are not quantitative but are qualitative, taking a discrete set of values.
- These are also called categorical predictors or factor variables.
- The `Credit` data set records variables for a number of credit card holders.
  - The response is `balance` (average credit card debt for each individual).
  - Quantitative predictors: `age`, `cards` (number of credit cards), `education` (years of education), `income` (in thousands of dollars), `limit` (credit limit), and `rating` (credit rating).
  - Qualitative variables: `gender`, `student` (student status), `status` (marital status), and `ethnicity` (Caucasian, African American (AA) or Asian).

- Example: investigate differences in credit card balance between males and females, ignoring the other variables. We create a new variable

$$X_i = \begin{cases} 1 & \text{if } i\text{-th person is female} \\ 0 & \text{if } i\text{-th person is male.} \end{cases}$$

- Resulting model:

$$Y_i = \beta_0 + \beta_1 X_i + \epsilon_i = \begin{cases} \beta_0 + \beta_1 + \epsilon_i & \text{if } i\text{-th person is female} \\ \beta_0 + \epsilon_i & \text{if } i\text{-th person is male.} \end{cases}$$

- With more than two levels, we create additional dummy variables. For example, for the ethnicity variable we create two dummy variables. The first could be

$$X_{i1} = \begin{cases} 1 & \text{if } i\text{-th person is Asian} \\ 0 & \text{if } i\text{-th person is not Asian,} \end{cases}$$

  and the second could be

$$X_{i2} = \begin{cases} 1 & \text{if } i\text{-th person is Caucasian} \\ 0 & \text{if } i\text{-th person is not Caucasian.} \end{cases}$$

- Both of these variables can be used:

$$\begin{aligned} Y_i &= \beta_0 + \beta_1 X_{i,1} + \beta_2 X_{i,2} + \epsilon_i \\ &= \begin{cases} \beta_0 + \beta_1 + \epsilon_i & \text{if } i\text{-th person is Asian} \\ \beta_0 + \beta_2 + \epsilon_i & \text{if } i\text{-th person is Caucasian} \\ \beta_0 + \epsilon_i & \text{if } i\text{-th person is AA.} \end{cases} \end{aligned}$$

- There will always be one fewer dummy variable than the number of levels. The level with no dummy variable is known as the baseline.

Interactions

- In our previous analysis of the `Advertising` data, we assumed that the effect on `sales` of increasing one advertising medium is independent of the amount spent on the other media.
- The average effect on `sales` of a one-unit increase in `TV` is always β_1, regardless of the amount spent on `radio`.
- But suppose that spending money on `radio` advertising actually increases the effectiveness of `TV` advertising, so that the slope term for `TV` should increase as `radio` increases.
- Model takes the form

$$\begin{aligned} \texttt{sales} =& \beta_0 + \beta_1 \times \texttt{TV} + \beta_2 \times \texttt{radio} + \beta_3 \times (\texttt{radio} \times \texttt{TV}) + \epsilon \\ =& \beta_0 + (\beta_1 + \beta_3 \times \texttt{radio}) \times \texttt{TV} + \beta_2 \times \texttt{radio} + \epsilon \end{aligned}$$

	Coefficient	Std. Error	t-statistic	p-value
`Intercept`	6.7502	0.248	27.23	< 0.0001
`TV`	0.0191	0.002	12.70	< 0.0001
`radio`	0.0289	0.009	3.24	0.0014
`TV` × `radio`	0.0011	0.000	20.73	< 0.0001

ISL Table 3.9

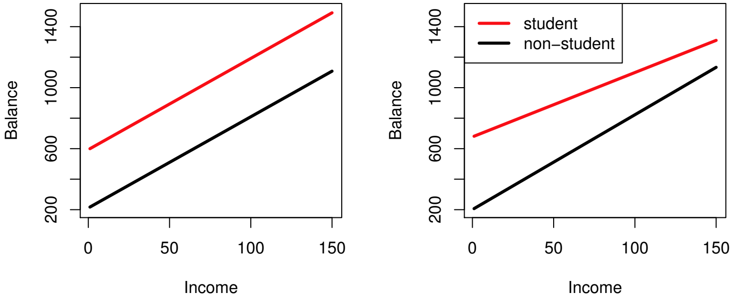
- The results suggest that interactions are important.
- The p-value for the interaction term `TV` × `radio` is extremely low, indicating that there is strong evidence for $\beta_3 \neq 0$.

- Consider the `Credit` data set, and suppose that we wish to predict `balance` using `income` (quantitative) and `student` (qualitative). Without an interaction term, the model takes the form

$$\begin{aligned}\texttt{balance}_i &\approx \beta_0 + \beta_1 \times \texttt{income}_i + \begin{cases}\beta_2 & \text{if } i\text{th person is a student}\\ 0 & \text{if } i\text{th person is not a student}\end{cases}\\ &= \beta_1 \times \texttt{income}_i + \begin{cases}\beta_0 + \beta_2 & \text{if } i\text{th person is a student}\\ \beta_0 & \text{if } i\text{th person is not a student.}\end{cases}\end{aligned}$$

- With interactions, it takes the form

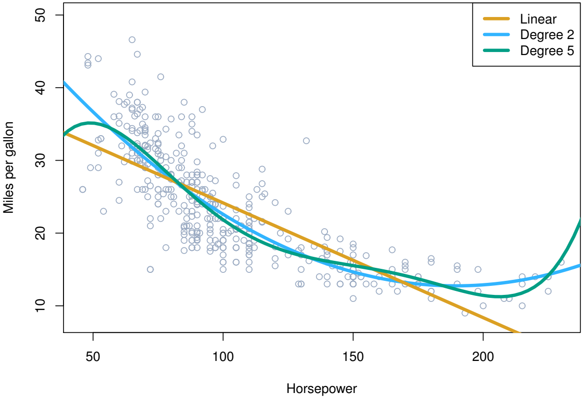
$$\begin{aligned}\texttt{balance}_i &\approx \beta_0 + \beta_1 \times \texttt{income}_i + \begin{cases}\beta_2 + \beta_3 \times \texttt{income}_i & \text{if student}\\ 0 & \text{if not student}\end{cases}\\ &= \begin{cases}(\beta_0 + \beta_2) + (\beta_1 + \beta_3) \times \texttt{income}_i & \text{if student}\\ \beta_0 + \beta_1 \times \texttt{income}_i & \text{if not student}\end{cases}\end{aligned}$$



ISL Figure 3.7

- Regression lines have different intercepts, as well as different slopes.

Non-linear effects of predictors



ISL Figure 3.8

- The `mpg` (gas mileage in miles per gallon) versus `horsepower` is shown for a number of cars in the `Auto` data set.

	Coefficient	Std. Error	t-statistic	p-value
`Intercept`	56.9001	1.8004	31.6	< 0.0001
`horsepower`	-0.4662	0.0311	-15.0	< 0.0001
`horsepower`^{2}	0.0012	0.0001	10.1	< 0.0001

ISL Table 3.10

- It seems clear that this relationship is in fact non-linear. A simple extension to the linear model is to include transformed predictors.
- A nonlinear model

$$\texttt{mpg} = \beta_0 + \beta_1 \times \texttt{horsepower} + \beta_2 \times \texttt{horsepower}^2 + \epsilon$$

may provide a better fit (lower R^2).