Introductory Econometrics

Lecture 16: Asymptotic normality

Instructor: Ma, Jun

Renmin University of China

November 8, 2021

Why do we need asymptotic normality?

- In the previous lectures, we have shown that the OLS estimator has an exact normal distribution when the errors are normally distributed
    - The same assumption is needed to show that the T statistic has a t-distribution and the F statistic has an F-distribution.
- In this lecture, we will argue that even when the errors are not normally distributed, the OLS estimator has an approximately normal distribution in large samples, provided that some additional conditions hold.
    - This property is used for hypothesis testing: in large samples, the T statistic has a standard normal distribution and the F statistic has a χ^2 distribution (approximately).

Asymptotic normality

- Let W_n be a sequence of random variables indexed by the sample size n.
  - Typically, W_n will be a function of some estimator. For example, we will have $W_n = \sqrt{n}\left(\hat{\beta}_n - \beta\right)$.
- We say that W_n has an asymptotically normal distribution if its CDF converges to a normal CDF.
- Let W be any random variable with a normal $\mathrm{N}\left(0, \sigma^2\right)$ distribution.We say that W_n has an asymptotically normal distribution if for all $x \in \mathbb{R}$:

$$F_n(x) = \Pr\left[W_n \leq x\right] \to \Pr\left[W \leq x\right] = F(x) \text{ as } n \to \infty.$$

  - We denote this as $W_n \to_d W$ or $W_n \to_d \mathrm{N}\left(0, \sigma^2\right)$.

Convergence in distribution

- Asymptotic normality is an example of the convergence in distribution.
- We say that a sequence of random variables W_n converges in distribution to W (denoted as $W_n \to_d W$) if the CDF of W_n converges to the CDF of W at all points where the CDF of W is continuous.
- Note that the convergence in distribution is convergence of the CDFs.

Central Limit Theorem (CLT)

- An example of convergence in distribution is a CLT.
- Let $X_1, \ldots, X_n$ be a sample of iid random variables such that $\mathrm{E}\left[X_i\right] = 0$ and $\mathrm{Var}\left[X_i\right] = \sigma^2 > 0$ (finite). Then, as $n \to \infty$,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i \to_d \mathrm{N}\left(0, \sigma^2\right).$$

CLT

- For the CLT we impose 3 assumptions: (1) iid; (2) Mean zero; (3) Finite variance different from zero.
- If $X_1, \ldots, X_n$ are iid but $\mathrm{E}[X_i] = \mu \neq 0$, then consider $X_i - \mu$. Since $\mathrm{E}[X_i - \mu] = 0$, we have

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - \mu) \to_d \mathrm{N}(0, \mathrm{Var}[X_i]).$$

Note that

$$\begin{aligned} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - \mu) &= \sqrt{n} \frac{1}{n} \sum_{i=1}^{n} (X_i - \mu) \\ &= \sqrt{n} \left(\frac{1}{n} \sum_{i=1}^{n} X_i - \frac{1}{n} \sum_{i=1}^{n} \mu \right) \\ &= \sqrt{n} \left(\bar{X}_n - \mu \right). \end{aligned}$$

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - \mu) = \sqrt{n} \left(\bar{X}_n - \mu \right).$$

- Thus, the CLT can be stated as

$$\sqrt{n} \left(\bar{X}_n - \mu \right) \to_d \mathrm{N}\left(0, \mathrm{Var}\left[X_i\right]\right).$$

- Note that by the LLN,

$$\bar{X}_n - \mu \to_p 0,$$

and

$$\mathrm{Var}\left[\sqrt{n} \left(\bar{X}_n - \mu \right)\right] = n \cdot \mathrm{Var}\left[\bar{X}_n\right] = n \frac{\mathrm{Var}\left[X_i\right]}{n} = \mathrm{Var}\left[X_i\right].$$

Properties

- Suppose that $W_n \to_d \mathrm{N}\left(0, \sigma^2\right)$ and $\theta_n \to_p \theta$. Then,
$$\theta_n W_n \to_d \theta \cdot \mathrm{N}\left(0, \sigma^2\right) =_d \mathrm{N}\left(0, \theta^2\sigma^2\right),$$
and
$$\theta_n + W_n \to_d \theta + \mathrm{N}\left(0, \sigma^2\right) =_d \mathrm{N}\left(\theta, \sigma^2\right).$$
- Suppose that $Z_n \to_d Z \sim \mathrm{N}(0, 1)$. Then,
$$Z_n^2 \to_d Z^2 =_d \chi_1^2.$$
- If $W_n \to_d c$ =constant, then $W_n \to_p c$.

Asymptotic normality of OLS

- Suppose that:
  1. The data $\{(Y_i, X_i) : i = 1, \ldots, n\}$ are iid.
  2. $Y_i = \beta_0 + \beta_1 X_i + U_i$, where $\mathrm{E}[U_i] = 0$.
  3. $\mathrm{E}[X_i U_i] = 0$.
  4. $0 < \mathrm{Var}[X_i] < \infty$.
  5. $0 < \mathrm{E}\left[(X_i - \mathrm{E}[X_i])^2 U_i^2\right] < \infty$ and $0 < \mathrm{E}\left[U_i^2\right] < \infty$.
- Let $\hat{\beta}_{1,n}$ be the OLS estimator of β_1. Then,

$$\sqrt{n}\left(\hat{\beta}_{1,n} - \beta_1\right) \to_d \mathrm{N}\left(0, \frac{\mathrm{E}\left[(X_i - \mathrm{E}[X_i])^2 U_i^2\right]}{(\mathrm{Var}[X_i])^2}\right).$$

- $V = \frac{\mathrm{E}\left[(X_i - \mathrm{E}[X_i])^2 U_i^2\right]}{(\mathrm{Var}[X_i])^2}$ is called the asymptotic variance of $\hat{\beta}_{1,n}$.

- Let $\overset{a}{\sim}$ denote approximately in large samples.
- The asymptotic normality

$$\sqrt{n}\left(\hat{\beta}_{1,n} - \beta_1\right) \to_d \mathrm{N}\left(0, V\right)$$

can be viewed as the following large-sample approximation:

$$\sqrt{n}\left(\hat{\beta}_{1,n} - \beta_1\right) \overset{a}{\sim} \mathrm{N}\left(0, V\right),$$

or

$$\hat{\beta}_{1,n} \overset{a}{\sim} \mathrm{N}\left(\beta_1, \frac{V}{n}\right).$$

Proof

Write

$$\hat{\beta}_{1,n} = \beta_1 + \frac{\sum_{i=1}^{n} \left(X_i - \bar{X}_n\right) U_i}{\sum_{i=1}^{n} \left(X_i - \bar{X}_n\right)^2}.$$

Now

$$\hat{\beta}_{1,n} - \beta_1 = \frac{\frac{1}{n} \sum_{i=1}^{n} \left(X_i - \bar{X}_n\right) U_i}{\frac{1}{n} \sum_{i=1}^{n} \left(X_i - \bar{X}_n\right)^2},$$

and

$$\sqrt{n} \left(\hat{\beta}_{1,n} - \beta_1\right) = \frac{\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(X_i - \bar{X}_n\right) U_i}{\frac{1}{n} \sum_{i=1}^{n} \left(X_i - \bar{X}_n\right)^2}.$$

$$\sqrt{n}\left(\hat{\beta}_{1,n}-\beta_1\right)=\frac{\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(X_i-\bar{X}_n\right)U_i}{\frac{1}{n}\sum_{i=1}^{n}\left(X_i-\bar{X}_n\right)^2}.$$

In the previous lecture, we established

$$\frac{1}{n}\sum_{i=1}^{n}\left(X_i-\bar{X}_n\right)^2\to_p \mathrm{Var}\left[X_i\right].$$

We will show that

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(X_i-\bar{X}_n\right)U_i\to_d \mathrm{N}\left(0,\mathrm{E}\left[\left(X_i-\mathrm{E}\left[X_i\right]\right)^2 U_i^2\right]\right),$$

so that

$$\begin{aligned}\sqrt{n}\left(\hat{\beta}_{1,n}-\beta_1\right)=\frac{\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(X_i-\bar{X}_n\right)U_i}{\frac{1}{n}\sum_{i=1}^{n}\left(X_i-\bar{X}_n\right)^2}&\to_d \frac{\mathrm{N}\left(0,\mathrm{E}\left[\left(X_i-\mathrm{E}\left[X_i\right]\right)^2 U_i^2\right]\right)}{\mathrm{Var}\left[X_i\right]}\\&=_d \mathrm{N}\left(0,\frac{\mathrm{E}\left[\left(X_i-\mathrm{E}\left[X_i\right]\right)^2 U_i^2\right]}{\left(\mathrm{Var}\left[X_i\right]\right)^2}\right).\end{aligned}$$

Proof of $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(X_i - \bar{X}_n\right) U_i \to_d \mathrm{N}\left(0, \mathrm{E}\left[\left(X_i - \mathrm{E}\left[X_i\right]\right)^2 U_i^2\right]\right)$

$$
\begin{aligned}
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(X_i - \bar{X}_n\right) U_i &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(X_i - \mathrm{E}\left[X_i\right] + \mathrm{E}\left[X_i\right] - \bar{X}_n\right) U_i \\
&= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(X_i - \mathrm{E}\left[X_i\right]\right) U_i + \left(\mathrm{E}\left[X_i\right] - \bar{X}_n\right) \frac{1}{\sqrt{n}} \sum_{i=1}^{n} U_i.
\end{aligned}
$$

We have

$$
\mathrm{E}\left[\left(X_i - \mathrm{E}\left[X_i\right]\right) U_i\right] = \mathrm{E}\left[X_i U_i\right] - \mathrm{E}\left[X_i\right] \mathrm{E}\left[U_i\right] = 0,
$$

and $0 < \mathrm{E}\left[\left(X_i - \mathrm{E}\left[X_i\right]\right)^2 U_i^2\right] < \infty$, so that by the CLT,

$$
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(X_i - \mathrm{E}\left[X_i\right]\right) U_i \to_d \mathrm{N}\left(0, \mathrm{E}\left[\left(X_i - \mathrm{E}\left[X_i\right]\right)^2 U_i^2\right]\right).
$$

It is left to show that

$$\left(\mathrm{E}\left[X_i\right]-\bar{X}_n\right)\frac{1}{\sqrt{n}}\sum_{i=1}^{n}U_i\to_p 0.$$

We have $\mathrm{E}\left[U_i\right]=0$ and $0<\mathrm{E}\left[U_i^2\right]<\infty$. Thus, by the CLT,

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}U_i\to_d \mathrm{N}\left(0,\mathrm{E}\left[U_i^2\right]\right).$$

By the LLN,

$$\mathrm{E}\left[X_i\right]-\bar{X}_n\to_p 0.$$

Now

$$\left(\mathrm{E}\left[X_i\right] - \bar{X}_n\right) \frac{1}{\sqrt{n}} \sum_{i=1}^{n} U_i \to_d 0 \cdot \mathrm{N}\left(0, \mathrm{E}\left[U_i^2\right]\right) = 0.$$

It follows that

$$\left(\mathrm{E}\left[X_i\right] - \bar{X}_n\right) \frac{1}{\sqrt{n}} \sum_{i=1}^{n} U_i \to_p 0.$$