Introductory Econometrics

Lecture 19: Instrumental variable estimation

Instructor: Ma, Jun

Renmin University of China

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Endogeneity

- In the linear regression model,

$$Y_i = \beta_0 + \beta_1 X_i + U_i$$

the condition for consistent estimation of β_1 by OLS is that X is exogenous:

$$\mathrm{Cov}\left[X_i, U_i\right] = 0.$$

- When

$$\mathrm{Cov}\left[X_i, U_i\right] \neq 0,$$

we say that the regressor X in endogenous.

- When the regressor is endogenous, the OLS estimator is inconsistent:

$$\hat{\beta}_{1,n} - \beta_1 = \frac{\frac{1}{n}\sum_{i=1}^{n}\left(X_i - \bar{X}_n\right)U_i}{\frac{1}{n}\sum_{i=1}^{n}\left(X_i - \bar{X}_n\right)^2} \to_p \frac{\mathrm{Cov}\left[X_i, U_i\right]}{\mathrm{Var}\left[X_i\right]} \neq 0.$$

Consequences of endogeneity

- The causal effect of X on Y is not estimated consistently

$$\hat{\beta}_{1,n} \to_p \beta_1 + \frac{\mathrm{Cov}\left[X_i, U_i\right]}{\mathrm{Var}\left[X_i\right]}.$$

- The effect can be over or under estimated depending on the sign of $\mathrm{Cov}\left[X_i, U_i\right]$.
- Tests and confidence intervals are invalid.

Sources of endogeneity

There are several possible sources of endogeneity:

1. Omitted explanatory variables.
2. Simultaneity.
3. Errors in variables.

All result in regressors correlated with the errors.

Omitted explanatory variables

- Suppose that the true model is

$$\log\left(Wage_i\right) = \beta_0 + \beta_1 Education_i + \beta_2 Ability_i + V_i,$$

where V_i is uncorrelated with $Education$ and $Ability$.

- Since $Ability$ is unobservable, the econometrician regresses $\log\left(Wage\right)$ against $Education$, and $\beta_2 Ability$ goes into the error part:

$$\begin{aligned} \log\left(Wage_i\right) &= \beta_0 + \beta_1 Education_i + U_i, \\ U_i &= \beta_2 Ability_i + V_i. \end{aligned}$$

- $Education$ is correlated with $Ability$: we can expect that $\mathrm{Cov}\left[Education_i, Ability_i\right] > 0, \beta_2 > 0$, and therefore

$$\mathrm{Cov}\left[Education_i, U_i\right] > 0.$$

Thus, OLS will overestimate the return to education.

Simultaneity

- Consider the following demand-supply system:

$$\begin{array}{ll} \text{Demand:} & Q^d = \beta_0^d + \beta_1^d P + U^d, \\ \text{Supply:} & Q^s = \beta_0^s + \beta_1^s P + U^s, \end{array}$$

  where: Q^d = quantity demanded, Q^s = quantity supplied, P = price.

- The quantity and price are determined simultaneously in the equilibrium:

$$Q^d = Q^s = Q.$$

- Note that Q^d and Q^s are not observed separately, we observe only the equilibrium values Q.

$$
\begin{aligned}
Q^d &= \beta_0^d + \beta_1^d P + U^d, \\
Q^s &= \beta_0^s + \beta_1^s P + U^s, \\
Q^d &= Q^s = Q.
\end{aligned}
$$

- Solving for P, we obtain

$$0 = \left(\beta_0^d - \beta_0^s\right) + \left(\beta_1^d - \beta_1^s\right) P + \left(U^d - U^s\right),$$

or

$$P = -\frac{\beta_0^d - \beta_0^s}{\beta_1^d - \beta_1^s} - \frac{U^d - U^s}{\beta_1^d - \beta_1^s}.$$

- Thus,

$$\mathrm{Cov}\left[P, U^d\right] \neq 0 \text{ and } \mathrm{Cov}\left[P, U^s\right] \neq 0.$$

The demand-supply equations cannot be estimated by OLS.

- Consider the following labour supply model for married women:

$$Hours_i = \beta_0 + \beta_1 Children_i + \text{Other Factors} + U_i,$$

  where $Hours$ = hours of work, $Children$ = number of children.
- It is reasonable to assume that women decide simultaneously how much time to devote to career and family.
- Thus, while we may be mainly interested in the effect of family size on labour supply, there is another equation:

$$Children_i = \gamma_0 + \gamma_1 Hours_i + \text{Other Factors} + V_i,$$

  and $Children$ and $Hours$ are determined simultaneously in an equilibrium.
- As a result, $\text{Cov}\left[Children_i, U_i\right] \neq 0$, and the effect of family size cannot be estimated by OLS.

Errors in variables

- Consider the following model:

$$Y_i = \beta_0 + \beta_1 X_i^* + V_i,$$

  where X_i^* is the true regressor.
- Suppose that X_i^* is not directly observable. Instead, we observe X_i that measures X_i^* with an error ε_i:

$$X_i = X_i^* + \varepsilon_i.$$

- Since X_i^* is unobservable, the econometrician has to regress Y_i against X_i.

$$
\begin{aligned}
X_i &= X_i^* + \varepsilon_i, \\
Y_i &= \beta_0 + \beta_1 X_i^* + V_i.
\end{aligned}
$$

- The model for Y_i as a function of X_i can be written as

$$
\begin{aligned}
Y_i &= \beta_0 + \beta_1 \left(X_i - \varepsilon_i\right) + V_i \\
&= \beta_0 + \beta_1 X_i + V_i - \beta_1 \varepsilon_i,
\end{aligned}
$$

or

$$
\begin{aligned}
Y_i &= \beta_0 + \beta_1 X_i + U_i, \\
U_i &= V_i - \beta_1 \varepsilon_i.
\end{aligned}
$$

$$
\begin{aligned}
Y_i &= \beta_0 + \beta_1 X_i + U_i, \\
U_i &= V_i - \beta_1 \varepsilon_i, \\
X_i &= X_i^* + \varepsilon_i.
\end{aligned}
$$

- We can assume that

$$
\text{Cov}\left[X_i^*, V_i\right] = \text{Cov}\left[X_i^*, \varepsilon_i\right] = \text{Cov}\left[\varepsilon_i, V_i\right] = 0.
$$

- However,

$$
\begin{aligned}
\text{Cov}\left[X_i, U_i\right] &= \text{Cov}\left[X_i^* + \varepsilon_i, V_i - \beta_1 \varepsilon_i\right] \\
&= \text{Cov}\left[X_i^*, V_i\right] - \beta_1 \text{Cov}\left[X_i^*, \varepsilon_i\right] \\
&\quad + \text{Cov}\left[\varepsilon_i, V_i\right] - \beta_1 \text{Cov}\left[\varepsilon_i, \varepsilon_i\right]
\end{aligned}
$$

- Thus, X_i is enodgenous and β_1 cannot be estimated by OLS.

Instrumental variable (IV)

- Consider

$$\begin{aligned} Y_i &= \beta_0 + \beta_1 X_i + U_i, \\ \mathrm{Cov}\left[X_i, U_i\right] &\neq 0. \end{aligned}$$

- Suppose that in addition, the econometrician observes another variable Z_i, called the instrumental variable, that satisfies the following conditions:
  1. The IV is exogenous: $\mathrm{Cov}\left[Z_i, U_i\right] = 0$.
  2. The IV determines the endogenous regressor: $\mathrm{Cov}\left[Z_i, X_i\right] \neq 0$.
- When an IV variable satisfying those conditions is available, it allows us to estimate the effect of X on Y consistently.

IV regression

$$
\begin{aligned}
Y_i &= \beta_0 + \beta_1 X_i + U_i, \\
\text{Cov}\left[Z_i, U_i\right] &= 0, \\
\text{Cov}\left[Z_i, X_i\right] &\neq 0.
\end{aligned}
$$

- Consider the following IV estimator of β_1:

$$
\hat{\beta}_{1,n}^{IV} = \frac{\sum_{i=1}^{n}\left(Z_i - \bar{Z}_n\right) Y_i}{\sum_{i=1}^{n}\left(Z_i - \bar{Z}_n\right) X_i}.
$$

- Write

$$
\begin{aligned}
\hat{\beta}_{1,n}^{IV} &= \frac{\sum_{i=1}^{n}\left(Z_i - \bar{Z}_n\right)\left(\beta_0 + \beta_1 X_i + U_i\right)}{\sum_{i=1}^{n}\left(Z_i - \bar{Z}_n\right) X_i} \\
&= \frac{\beta_0 \sum_{i=1}^{n}\left(Z_i - \bar{Z}_n\right) + \beta_1 \sum_{i=1}^{n}\left(Z_i - \bar{Z}_n\right) X_i + \sum_{i=1}^{n}\left(Z_i - \bar{Z}_n\right) U_i}{\sum_{i=1}^{n}\left(Z_i - \bar{Z}_n\right) X_i} \\
&= \beta_1 + \frac{\sum_{i=1}^{n}\left(Z_i - \bar{Z}_n\right) U_i}{\sum_{i=1}^{n}\left(Z_i - \bar{Z}_n\right) X_i}.
\end{aligned}
$$

Consistency of the IV estimator

$$\text{Cov}\left[Z_i, U_i\right] = 0 \tag{1}$$

$$\text{Cov}\left[Z_i, X_i\right] \neq 0. \tag{2}$$

- Using the LLN (and under some additional technical conditions), (1) implies that

$$\frac{1}{n}\sum_{i=1}^{n}\left(Z_i - \bar{Z}_n\right)U_i \to_p \text{Cov}\left[Z_i, U_i\right],$$

  and (1) implies that

$$\frac{1}{n}\sum_{i=1}^{n}\left(Z_i - \bar{Z}_n\right)X_i \to_p \text{Cov}\left[Z_i, X_i\right].$$

- The IV estimator is consistent if (1) and (2) are true:

$$\hat{\beta}_{1,n}^{IV} = \beta_1 + \frac{\frac{1}{n}\sum_{i=1}^{n}\left(Z_i - \bar{Z}_n\right)U_i}{\frac{1}{n}\sum_{i=1}^{n}\left(Z_i - \bar{Z}_n\right)X_i} \to_p \beta_1 + \frac{\text{Cov}\left[Z_i, U_i\right]}{\text{Cov}\left[Z_i, X_i\right]} = \beta_1 + \frac{0}{\text{Cov}\left[Z_i, X_i\right]} = \beta_1.$$

Natural experiments

- Theoretically, the causal effect can be estimated from controlled experiments:
  - To estimate the return to education, select a random sample of children, randomly assign how many years of education they should have, and measure their income several years after the graduation.
  - To estimate the effect of family size on labor supply, select a random sample of parents and randomly assign how many children they should have, and measure their labor market outcomes.
  - Such an approach is infeasible due to a high cost and/or ethical reasons.
- Natural experiments: Use the random variation in the variable of interest to estimate the causal effect.

Example: Compulsory schooling laws and return to education

- Angrist and Krueger, 1991, *QJE*, suggested using school start age policy to estimate β_1 in $\log\left(Wage_i\right) = \beta_0 + \beta_1 Education_i + \beta_2 Ability_i + V_i$.
- We need to find an IV variable Z such that $\text{Cov}\left[Ability_i, Z_i\right] = 0$ and $\text{Cov}\left[Education_i, Z_i\right] \neq 0$.
- They argue that due to compulsory schooling laws, the season of birth variable satisfies the IV conditions:
  - A child has to attend the school until he reaches a certain drop-out age.
  - Students born in the first quarter of the year, reach the legal drop-out age before their classmates who were born later in the year.
  - The quarter of birth dummy variable is correlated with education.
  - The quarter of birth is uncorrelated with ability.

Example: Sibling-sex composition and labor supply

- Angrist and Evans, 1998, *AER*, argue that the parents' preferences for a mixed sibling-sex composition can be used to estimate β_1 in $Hours_i = \beta_0 + \beta_1 Children_i + \ldots + U_i$.
- We need to find an IV Z such that $\text{Cov}\left[U_i, Z_i\right] = 0$ and $\text{Cov}\left[Children_i, Z_i\right] \neq 0$.
- Consider a dummy variable that takes on the value one if the sex of the second child matches the sex of the first child.
  - If the parents prefer a mixed sibling-sex composition, they are more likely to have another child if their first two children are of the same sex.
  - The same-sex dummy is correlated with the number of children.
  - Since sex mix is randomly determined, the same sex dummy is exogenous.

The asymptotic distribution of the IV estimator

$$
\begin{aligned}
\hat{\beta}_{1,n}^{IV} &= \beta_1 + \frac{\sum_{i=1}^{n}\left(Z_i - \bar{Z}_n\right)U_i}{\sum_{i=1}^{n}\left(Z_i - \bar{Z}_n\right)X_i}, \\
Cov\left[Z_i, U_i\right] &= 0, \\
Cov\left[Z_i, X_i\right] &\neq 0.
\end{aligned}
$$

- Write

$$
\sqrt{n}\left(\hat{\beta}_{1,n}^{IV} - \beta_1\right) = \frac{\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(Z_i - \bar{Z}_n\right)U_i}{\frac{1}{n}\sum_{i=1}^{n}\left(Z_i - \bar{Z}_n\right)X_i} \to_d \frac{\mathrm{N}\left(0, \mathrm{E}\left[\left(Z_i - \mathrm{E}\left[Z_i\right]\right)^2 U_i^2\right]\right)}{\mathrm{Cov}\left[Z_i, X_i\right]}.
$$

- Thus,

$$
\sqrt{n}\left(\hat{\beta}_{1,n}^{IV} - \beta_1\right) \to_d \mathrm{N}\left(0, V^{IV}\right), \text{ where}
$$

$$
V^{IV} = \frac{\mathrm{E}\left[\left(Z_i - \mathrm{E}\left[Z_i\right]\right)^2 U_i^2\right]}{\left(\mathrm{Cov}\left[Z_i, X_i\right]\right)^2}.
$$

Variance estimation

$$\sqrt{n}\left(\hat{\beta}^{IV}_{1,n} - \beta_1\right) \to_d \mathrm{N}\left(0, V^{IV}\right), \text{ where } V^{IV} = \frac{\mathrm{E}\left[(Z_i - \mathrm{E}\left[Z_i\right])^2 U_i^2\right]}{(\mathrm{Cov}\left[Z_i, X_i\right])^2}.$$

- Let $\hat{\beta}^{IV}_{0,n} = \bar{Y}_n - \hat{\beta}^{IV}_{1,n} \cdot \bar{X}_n$. Let $\hat{U}_i = Y_i - \hat{\beta}^{IV}_{0,n} - \hat{\beta}^{IV}_{1,n} X_i$.
- Estimate V^{IV}

$$\hat{V}^{IV}_n = \frac{\frac{1}{n}\sum_{i=1}^n \left(Z_i - \bar{Z}_n\right)^2 \hat{U}_i^2}{\left(\frac{1}{n}\sum_{i=1}^n \left(Z_i - \bar{Z}_n\right) X_i\right)^2}.$$

- In finite samples, we use the following approximation:

$$\hat{\beta}^{IV}_{1,n} \overset{a}{\sim} \mathrm{N}\left(\beta_1, \frac{\hat{V}^{IV}_n}{n}\right).$$