

Introductory Econometrics

Lecture 9: Hypothesis testing for linear regression models

Instructor: Ma, Jun

Renmin University of China

October 1, 2021

Null and alternative hypotheses

- ▶ Usually, we have two competing hypotheses, and we want to draw a conclusion, based on the data, as to which of the hypotheses is true.
- ▶ Null hypothesis, denoted as H_0 : A hypothesis that is held to be true, unless the data provides a sufficient evidence against it.
- ▶ Alternative hypothesis, denoted as H_1 : A hypothesis against which the null is tested. It is held to be true if the null is found false.
- ▶ The two hypotheses are not treated symmetrically. H_0 is favored in the sense that it is only rejected if there is strong evidence against it.
- ▶ H_0 summarizes our prior belief about the state of the world.
- ▶ H_1 corresponds to our belief about how H_0 could be wrong.
- ▶ The two hypotheses must be disjoint: it should be the case that either H_0 is true or H_1 but never together simultaneously.

Heads or Tails?

- ▶ We believe that a coin is fair, unless we see enough evidence against it.
- ▶ In the coin tossing example, let p be the probability of “Heads”. Our null hypothesis could be $H_0 : p = \frac{1}{2}$ and our alternative hypothesis could be $H_1 : p \neq \frac{1}{2}$.
- ▶ We may believe that the coin is fair, but suspect that it is biased towards heads if it is biased. In this case, we may pick $H_0 : p = \frac{1}{2}$ and $H_1 : p > \frac{1}{2}$.

Hypothesis testing as stochastic proof by contradiction

- ▶ Usually, the econometrician has to carry the “burden of proof” and the case that he is interested in is stated as H_1 .
- ▶ The econometrician has to prove that his assertion (H_1) is true by showing that the data rejects H_0 .
- ▶ As we perform proof by contradiction, we assume “something” we want to prove that it contradicts to itself.
- ▶ In the hypothesis testing, we assume the “null hypothesis” being true and we want to prove that our data contradicts to the hypothesis.

Hypothesis testing procedure

- ▶ Suppose we are concerned with a parameter $\theta \in \mathbb{R}$ (e.g. the population mean of some random variable for which we have repeated observations).
- ▶ We consider hypotheses like $H_0 : \theta \in I_0$ and $H_1 : \theta \in I_1$ with $I_0 \cap I_1 = \emptyset$. $I_0 \cup I_1$ is the maintained hypothesis.
- ▶ The econometrician has to choose between H_0 and H_1 .
- ▶ We consider a test statistic T , which is a function of the data. T should be informative about the value of θ . T typically has a known distribution under H_0 .
- ▶ Then we choose a significance level α , with $0 < \alpha < 1$. It is also called the size of the test. By convention, α is chosen to be a small number.
- ▶ α can be interpreted as the probability of a “very small-probability” event: How small a chance should be considered small?

Hypothesis testing procedure

- ▶ Then one rejects H_0 if the test statistic falls into a critical region Γ_α . A critical region is constructed by taking into account the probability of making a wrong decision, i.e. it typically depends on α . This is usually a set of possible values of the test statistic that contains the test statistic with probability α , under H_0 , i.e. $\Pr[T \in \Gamma_\alpha] = \alpha$ under H_0 . Notice that this requires knowledge about the distribution of T under H_0 .
- ▶ If the test statistic assumes a value in the critical region, $T \in \Gamma_\alpha$, this is considered to be strong evidence against H_0 . In this case, we reject H_0 in favor of H_1 . Otherwise, we fail to reject H_0 .
- ▶ If α is small and $T \in \Gamma_\alpha$, it means that the sample is unlikely if H_0 is true and so we have evidence against it, there are two possibilities: we observed a very small-probability event or our assumption (H_0) is wrong.

Hypothesis testing as stochastic proof by contradiction

- ▶ If we accept the principle that a very small-probability event cannot happen in the current state of the world, we accept the second reasoning: our assumption (H_0) is wrong.
- ▶ If $T \notin \Gamma_\alpha$, there is no “significant” contradiction between data and the null hypothesis. On the other hand, if $T \in \Gamma_\alpha$, there is a “significant” contradiction and we reject H_0 like what we conclude in proof by contradiction.
- ▶ Notice that “reject H_0 ” and “fail to reject H_0 ” are used for conclusions drawn from tests. Such terminology reflects the asymmetry of H_0 and H_1 .
- ▶ We do not say “accept H_0 ” instead of “fail to reject H_0 ” since if we find $T \notin \Gamma_\alpha$, we did not “prove” anything and our test result is non-conclusive, since we did not find a contradiction.
- ▶ The decision depends on the significance level α : larger values of α correspond to bigger critical regions Γ_α . It is easier to reject the null for larger values of α .

A simple example

- ▶ Consider a normal population with mean μ and variance σ^2 and a random sample from the population. Suppose we know σ^2 for now.
- ▶ Consider a one-sided test: $H_0 : \mu = 0$ against $H_1 : \mu > 0$.
- ▶ Consider

$$T = \frac{\bar{X}}{\sigma/\sqrt{n}},$$

which is a $N(0, 1)$ random variable under H_0 . H_1 is more likely to be true if T is large.

- ▶ Consider $\Gamma_\alpha = [z_{1-\alpha}, \infty)$, where $z_{1-\alpha}$ is the $(1 - \alpha)$ -th quantile of $N(0, 1)$.
- ▶ Under H_0 , $\Pr[T \in \Gamma_\alpha] = \alpha$. And notice that Γ_α is a sub-interval of $\Gamma_{\alpha'}$ if $\alpha < \alpha'$.

Summary of the steps

- ▶ The following are the steps of the hypothesis testing:
 1. Specify H_0 and H_1 .
 2. Choose the significance level α .
 3. Define a decision rule (critical region).
 4. Perform the test using the data: given the data compute the test statistic and see if it falls into the critical region.

p -Value

- ▶ Given a realization t of the test statistic T , we can compute the lowest significance level consistent with rejecting H_0 . This is called the p -value:

$$p\text{-value} = \min \{ 0 < \alpha < 1 : t \in \Gamma_\alpha \} .$$

- ▶ The p -value could be viewed as a measure of contradiction. The smaller the p -value is, the larger the contradiction is. In a probabilistic model, we use the way of reasoning in proof by contradiction with a “measure” of contradiction based on data to carry out the similar reasoning.
- ▶ Now if the p -value is smaller than our tolerance (significance level), then we reject null hypothesis like what we conclude in proof by contradiction.

The simple example

- ▶ In the simple example, the critical region is $\Gamma_\alpha = [z_{1-\alpha}, \infty)$. Suppose we have a realization t of T .
- ▶ Suppose we find that $z_{1-p} = t$. We would reject H_0 for all significance levels $\alpha \geq p$. p is the realized p -value.

Errors

- There are two types of errors that the econometrician can make:

Decision	Truth	
	H_0	H_1
	H_0	Type II error
	H_1	Type I error
		✓

- Type I error is the error of rejecting H_0 when H_0 is true.

$$\Pr(\text{Type I error}) = \Pr(\text{reject } H_0 \mid H_0 \text{ is true}) = \alpha.$$

- Type II error is the error of not rejecting H_0 when H_1 is true.

Errors

- ▶ The real line is split into two regions: acceptance region and rejection region (critical region).
 - ▶ When $T \notin \Gamma_\alpha$, we accept H_0 (and risk making a Type II error).
 - ▶ When $T \in \Gamma_\alpha$, we reject H_0 (and risk making a Type I error).
 - ▶ Unfortunately, the probabilities of Type I and II errors are inversely related.
 - ▶ By decreasing the probability of Type I error α , one makes the critical region smaller, which increases the probability of the Type II error. Thus it is impossible to make both errors arbitrary small.
- ▶ We require that the test we carry out has to be valid: the probability of type-I error ($\Pr[T \in \Gamma_\alpha]$ under H_0) must be α .
- ▶ We want the probability of a type-II error to be as small as possible for a given probability of a type-I error.

Power of tests

- ▶ Power of a test:

$$1 - \Pr(\text{Type II error}) = 1 - \Pr(\text{Accept } H_0 \mid H_0 \text{ is false}) .$$

- ▶ We want the power of a test to be as large as possible, for a given significance level.
- ▶ We may not know the distribution of the test statistic under H_1 . The distribution typically depends on θ .
- ▶ The power function $\pi(\theta)$ of the test is the probability that H_0 is rejected as a function of the true parameter value θ .

The simple example

- For any given value of μ , define

$$T_{\mu} = T - \frac{\mu}{\sigma/\sqrt{n}} = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}},$$

which is always $N(0, 1)$ if the true mean is μ .

- We reject the test when $T \geq z_{1-\alpha}$, which holds if and only if

$$T_{\mu} = T - \frac{\mu}{\sigma/\sqrt{n}} \geq z_{1-\alpha} - \frac{\mu}{\sigma/\sqrt{n}}.$$

It follows for this simple example, that the power function is

$$\pi(\mu) = 1 - \Phi\left(z_{1-\alpha} - \frac{\sqrt{n}\mu}{\sigma}\right).$$

- Notice that $\pi(0) = 1 - \Phi(z_{1-\alpha}) = \alpha$, which is simply the significance level.

The simple example

- ▶ We notice that $\pi(\mu)$ is smaller for all μ if the significance level α is smaller (and hence $z_{1-\alpha}$ is larger). This reflects the trade-off between type-I error and type-II error probabilities: we cannot reduce both simultaneously.
- ▶ $\pi(\mu)$ is increasing in μ . For μ 's that are farther away from 0, the test can detect such deviation at a higher probability.
- ▶ As $\mu \rightarrow \infty$, the power converges to 1. The test is very likely to reject H_0 if the true mean is very large.
- ▶ $\pi(\mu)$ increases with the sample size n . The test can detect falseness of H_0 at a higher probability if our sample contains more information.

The simple example

- ▶ We consider an alternative critical region: $\tilde{\Gamma}_\alpha = (-\infty, -z_{1-\alpha}]$. Notice that under H_0 , $\Pr [T \in \tilde{\Gamma}_\alpha] = \alpha$, so this is a valid test.
- ▶ We reject the test when $T \leq -z_{1-\alpha}$, which holds if and only if

$$T_\mu = T - \frac{\mu}{\sigma/\sqrt{n}} \leq -z_{1-\alpha} - \frac{\mu}{\sigma/\sqrt{n}}.$$

- ▶ So the power function is

$$\tilde{\pi}(\mu) = \Phi \left(-z_{1-\alpha} - \frac{\sqrt{n}\mu}{\sigma} \right).$$

- ▶ $\tilde{\pi}(\mu)$ is decreasing in both μ and n , which is undesirable.
- ▶ For all $\mu > 0$ (remember μ is the true mean), the power is smaller than α . At the same significance level, the power of the alternative critical region is much lower.

Linear model assumptions

- ▶ β_1 is unknown, and we have to rely on its OLS estimator $\hat{\beta}_1$.
- ▶ We need to know the distribution of $\hat{\beta}_1$ or of its certain functions.
- ▶ We will assume that the assumptions of the Normal Classical Linear Regression model are satisfied:
 1. $Y_i = \beta_0 + \beta_1 X_i + U_i, \quad i = 1, \dots, n.$
 2. $E[U_i | X_1, \dots, X_n] = 0$ for all i 's.
 3. $E[U_i^2 | X_1, \dots, X_n] = \sigma^2$ for all i 's.
 4. $E[U_i U_j | X_1, \dots, X_n] = 0$ for all $i \neq j$.
 5. U 's are jointly normally distributed conditional on X 's.
- ▶ Recall that, in this case, conditionally on X 's:

$$\hat{\beta}_1 \sim N(\beta_1, \text{Var}[\hat{\beta}_1]), \text{ where } \text{Var}[\hat{\beta}_1] = \frac{\sigma^2}{\sum_{i=1}^n (X_i - \bar{X})^2}.$$

Two-sided tests

- ▶ For $Y_i = \beta_0 + \beta_1 X_i + U_i$, consider testing

$$H_0 : \beta_1 = \beta_{1,0},$$

against

$$H_1 : \beta_1 \neq \beta_{1,0}.$$

- ▶ β_1 is the true unknown value of the slope parameter.
- ▶ $\beta_{1,0}$ is a known number specified by the econometrician. (For example $\beta_{1,0}$ is zero if you want to test $\beta_1 = 0$).
- ▶ Such a test is called two-sided because the alternative hypothesis H_1 does not specify in which direction β_1 can deviate from the asserted value $\beta_{1,0}$.

Two-sided tests when σ^2 is known (infeasible test)

- ▶ Suppose for a moment that σ^2 is known.
- ▶ Consider the following test statistic:

$$T = \frac{\hat{\beta}_1 - \beta_{1,0}}{\sqrt{\text{Var} [\hat{\beta}_1]}}, \text{ where } \text{Var} [\hat{\beta}_1] = \frac{\sigma^2}{\sum_{i=1}^n (X_i - \bar{X})^2}.$$

- ▶ Consider the following decision rule (test):

$$\text{Reject } H_0 : \beta_1 = \beta_{1,0} \text{ when } |T| > z_{1-\alpha/2},$$

where $z_{1-\alpha/2}$ is the $(1 - \alpha/2)$ quantile of the standard normal distribution (critical value).

Test validity and power

- We need to establish that:

1. The test is valid, where the validity of a test means that it has correct size or $\Pr(\text{Type I error}) = \alpha$: if $\beta_1 = \beta_{1,0}$

$$\Pr[|T| > z_{1-\alpha/2}] = \alpha.$$

2. The test has power: when $\beta_1 \neq \beta_{1,0}$ (H_0 is false), the test rejects H_0 with probability that exceeds α : if $\beta_1 \neq \beta_{1,0}$

$$\Pr[|T| > z_{1-\alpha/2}] > \alpha.$$

- We want $\Pr[|T| > z_{1-\alpha/2}]$ when $\beta_1 \neq \beta_{1,0}$ to be as large as possible.
- Note that $\Pr[|T| > z_{1-\alpha/2}]$ when $\beta_1 \neq \beta_{1,0}$ depends on the true value β_1 .

The distribution of T when σ^2 is known (infeasible test)

- Write

$$\begin{aligned} T &= \frac{\hat{\beta}_1 - \beta_{1,0}}{\sqrt{\text{Var} [\hat{\beta}_1]}} = \frac{\hat{\beta}_1 - \beta_1 + \beta_1 - \beta_{1,0}}{\sqrt{\text{Var} [\hat{\beta}_1]}} \\ &= \frac{\hat{\beta}_1 - \beta_1}{\sqrt{\text{Var} [\hat{\beta}_1]}} + \frac{\beta_1 - \beta_{1,0}}{\sqrt{\text{Var} [\hat{\beta}_1]}}. \end{aligned}$$

- Under our assumptions and conditionally on X 's:

$$\hat{\beta}_1 \sim N(\beta_1, \text{Var} [\hat{\beta}_1]), \text{ or } \frac{\hat{\beta}_1 - \beta_1}{\sqrt{\text{Var} [\hat{\beta}_1]}} \sim N(0, 1).$$

- We have that conditionally on X 's: $T \sim N\left(\frac{\beta_1 - \beta_{1,0}}{\sqrt{\text{Var} [\hat{\beta}_1]}}, 1\right).$

Validity when σ^2 is known (infeasible test)

- We have that

$$T \sim \mathcal{N} \left(\frac{\beta_1 - \beta_{1,0}}{\sqrt{\text{Var} [\hat{\beta}_1]}}, 1 \right).$$

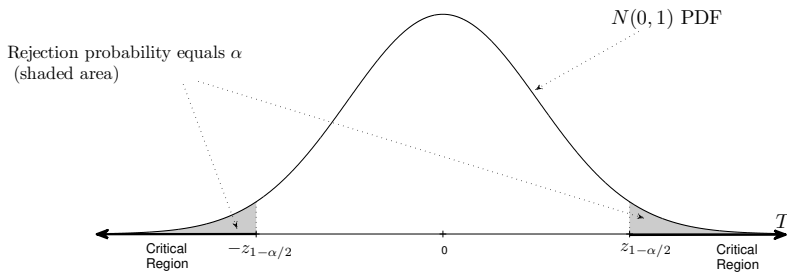
- When $H_0 : \beta_1 = \beta_{1,0}$ is true, $T \stackrel{H_0}{\sim} \mathcal{N}(0, 1)$.
- We reject H_0 when

$$|T| > z_{1-\alpha/2} \Leftrightarrow T > z_{1-\alpha/2} \text{ or } T < -z_{1-\alpha/2}.$$

- Let $Z \sim \mathcal{N}(0, 1)$.

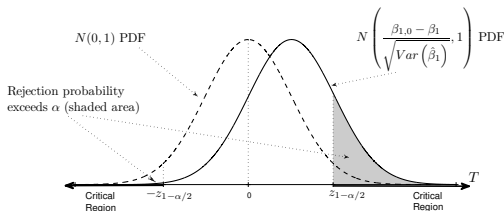
$$\begin{aligned} \Pr(\text{Reject } H_0 \mid H_0 \text{ is true}) &= \Pr[Z > z_{1-\alpha/2}] + \Pr[Z < -z_{1-\alpha/2}] \\ &= \alpha/2 + \alpha/2 = \alpha \end{aligned}$$

The distribution of T when σ^2 is known (infeasible test)



Power of the test when σ^2 is known (infeasible test)

- ▶ Under H_1 , $\beta_1 - \beta_{1,0} \neq 0$ and the distribution of T is not centered zero: $T \sim N\left(\frac{\beta_1 - \beta_{1,0}}{\sqrt{\text{Var}[\hat{\beta}_1]}}, 1\right)$.
- ▶ When $\beta_1 - \beta_{1,0} > 0$:



- ▶ Rejection probability exceeds α under H_1 : power increases with the distance from H_0 ($|\beta_{1,0} - \beta_1|$) and decreases with $\text{Var}[\hat{\beta}_1]$.

The two-sided t -test

- ▶ We are testing $H_0 : \beta_1 = \beta_{1,0}$ against $H_1 : \beta_1 \neq \beta_{1,0}$.
- ▶ When σ^2 is unknown, we replace it with $s^2 = \frac{1}{n-2} \sum_{i=1}^n \hat{U}_i^2$.
- ▶ The t -statistic:

$$T = \frac{\hat{\beta}_1 - \beta_{1,0}}{\sqrt{\widehat{\text{Var}}[\hat{\beta}_1]}} = \frac{\hat{\beta}_1 - \beta_{1,0}}{\sqrt{\frac{s^2}{\sum_{i=1}^n (X_i - \bar{X})^2}}}.$$

- ▶ We also replace the standard normal critical values $z_{1-\alpha/2}$ with the t_{n-2} critical values $t_{n-2, 1-\alpha/2}$.
However, for large n , $t_{n-2, 1-\alpha/2} \approx z_{1-\alpha/2}$.
- ▶ The two-sided t -test:

Reject H_0 when $|T| > t_{n-2, 1-\alpha/2}$.

The two-sided p -value

- ▶ The decision to accept or reject H_0 depends on the critical value $t_{n-2, 1-\alpha/2}$.
- ▶ If $\alpha_1 > \alpha_2$ then $t_{1-\alpha_1/2} < t_{1-\alpha_2/2}$.
- ▶ Thus, it is easier to reject H_0 with the significance level α_1 since it corresponds to a smaller acceptance region.
- ▶ p -value is the smallest significance level α for which we can reject H_0 .

The two-sided p -value

- ▶ In order to find p -value:
 1. Compute T .
 2. Find τ such that $|T| = t_{n-2, 1-\tau}$.
 3. The p -value $= \tau \times 2$.
- ▶ Note that for all $\alpha > p$ -value,

$$|T| = t_{n-2, 1-(p\text{-value})/2} > t_{n-2, 1-\alpha/2}$$

and we will reject H_0 .

- ▶ For all $\alpha \leq p$ -value,

$$|T| = t_{n-2, 1-(p\text{-value})/2} \leq t_{n-2, 1-\alpha/2}$$

and we will accept H_0 .

Example of p -value calculation

Suppose a regression with 19 observations produced the following output:

y	Coef.	Std. Err.	t	P> t	[95% Conf. Interval]	
x	-.6725304	.5804943	-1.16	0.263	-1.897266	.5522055
_cons	10.18197	.2509365	40.58	0.000	9.652542	10.7114

- ▶ Here, $\hat{\beta}_1 = -0.6725$, $\beta_{1,0} = 0$, and in the 4th column $t = -0.6725/0.5804 = -1.16$.
 - ▶ Thus, $|T| = 1.16$ and $df=17$.
 - ▶ From the t -table, the closest critical value is $t_{17,1-0.15} = 1.069$.
(The probability that a random variable with t_{17} -distribution lies on the right of 1.16 is ≈ 0.15 .)
 - ▶ The p -value is then $\approx 0.15 \times 2 = 0.300$.

Stata

- ▶ We can compute critical values and p -values using Stata instead of using the tables.
- ▶ To compute standard normal critical values use:

`display invnormal(τ),`

where τ is a number between 0 and 1.

- ▶ For example: `display invnormal(1-0.05/2)` produces 1.959964.
- ▶ For t critical values use

`display invttail(df , τ),`

where df is the number of degrees of freedom and τ is a number between 0 and 1.

Note that here τ is the right-tail probability!

- ▶ For example, `display invttail(62,0.05/2)` produces 1.9989715.

Stata

- ▶ To compute two-sided normal p -values use:

display 2 * (1-normal (T)) .

- ▶ For example, display 2*(1-normal(1.96)) produces 0.04999579.
- ▶ To compute two-sided t -distribution p -values, use

display 2* (ttail(df , T)) ,

Note that ttail gives the right tail probabilities!

- ▶ For example, display 2*(ttail(62, 1.96)) produces 0.05449415.

Example

rent	Coef.	Std. Err.	t	P> t	[95% Conf. Interval]	
avginc	.01158	.0013084	8.85	0.000	.0089646	.0141954
_cons	148.7764	32.09787	4.64	0.000	84.6137	212.9392

- ▶ Stata report the t -statistics and the p -value for $H_0 : \beta = 0$.
- ▶ To test H_0 whether the coefficient of AvgInc is zero :
 $T = 0.01158/0.0013084 = 8.85$.
- ▶ The p -value is extremely close to zero, (display $2*(\text{ttail}(62, 8.85))$ gives 1.345×10^{-12}), so for all reasonable significance levels α , we reject H_0 that the coefficient of AvgInc is zero.
- ▶ AvgInc is a statistically significant regressor.

Example

rent	Coef.	Std. Err.	t	P> t	[95% Conf. Interval]	
avginc	.01158	.0013084	8.85	0.000	.0089646	.0141954
_cons	148.7764	32.09787	4.64	0.000	84.6137	212.9392

- ▶ Consider now testing H_0 that the coefficient of AvgInc is 0.009 against the alternative that it is different from 0.009.
- ▶ $T = (0.01158 - 0.009) / 0.0013084 \approx 1.97$.
- ▶ At 5% significance level, $t_{62,0.975} \approx 1.999 > T$ and we accept H_0 .
- ▶ At 10% significance level, $t_{62,0.95} \approx 1.67 < T$ and we reject H_0 .
- ▶ The two sided p -value is $2*(\text{ttail}(62, 1.97)) \implies \approx 0.053$.
- ▶ For $\alpha \leq 0.053$ we will accept H_0 and for $\alpha > 0.053$ we will reject H_0 .

Confidence intervals and hypothesis testing

- ▶ There is one-to-one correspondence between confidence intervals and hypothesis testing.
- ▶ We cannot reject $H_0 : \beta_1 = \beta_{1,0}$ against a two-sided alternative if $|T| \leq t_{n-2, 1-\alpha/2}$ or if and only if:

$$-t_{n-2, 1-\alpha/2} \leq \frac{\hat{\beta}_1 - \beta_{1,0}}{\sqrt{\widehat{\text{Var}}[\hat{\beta}_1]}} \leq t_{n-2, 1-\alpha/2}$$

$$\Longleftrightarrow$$

$$\hat{\beta}_1 - t_{n-2, 1-\alpha/2} \sqrt{\widehat{\text{Var}}[\hat{\beta}_1]} \leq \beta_{1,0} \leq \hat{\beta}_1 + t_{n-2, 1-\alpha/2} \sqrt{\widehat{\text{Var}}[\hat{\beta}_1]}$$

$$\Longleftrightarrow$$

$$\beta_{1,0} \in CI_{1-\alpha}.$$

- ▶ Thus, for any $\beta_{1,0} \in CI_{1-\alpha}$, we cannot reject $H_0 : \beta_1 = \beta_{1,0}$ against $H_1 : \beta_1 \neq \beta_{1,0}$ at significance level α .

Example

```
. regress rent avginc
```

Source	SS	df	MS	Number of obs = 64		
Model	347069.249	1	347069.249	F(1, 62) = 78.34		
Residual	274693.188	62	4430.53529	Prob > F = 0.0000		
Total	621762.438	63	9869.24504	R-squared = 0.5582		
				Adj R-squared = 0.5511		
				Root MSE = 66.562		

rent	Coef.	Std. Err.	t	P> t	[95% Conf. Interval]	
avginc	.01158	.0013084	8.85	0.000	.0089646	.0141954
_cons	148.7764	32.09787	4.64	0.000	84.6137	212.9392

- ▶ The 95% confidence interval for the coefficient of AvgInc is [0.0089646,0.0141954].
- ▶ A significance level 5% test of $H_0 : \beta_1 = \beta_{1,0}$ against $H_1 : \beta_1 \neq \beta_{1,0}$ will not reject H_0 if $\beta_{1,0} \in [0.0089646, 0.0141954]$.

One-sided tests

- ▶ Consider testing $H_0 : \beta_1 \leq \beta_{1,0}$ against $H_1 : \beta_1 > \beta_{1,0}$.
- ▶ It is reasonable to reject H_0 when $\hat{\beta}_1 - \beta_{1,0}$ is large and positive or when

$$T = \frac{\hat{\beta}_1 - \beta_{1,0}}{\sqrt{\widehat{\text{Var}}[\hat{\beta}_1]}} > c_{1-\alpha}$$

where $c_{1-\alpha}$ is a positive constant.

- ▶ The null hypothesis H_0 is composite. The probability of rejection under H_0 depends on β_1 .
- ▶ We pick the critical value $c_{1-\alpha}$ so that

$$\Pr \left[\frac{\hat{\beta}_1 - \beta_{1,0}}{\sqrt{\widehat{\text{Var}}[\hat{\beta}_1]}} > c_{1-\alpha} \mid \beta_1 \leq \beta_{1,0} \right] \leq \alpha$$

for all $\beta_1 \leq \beta_{1,0}$.

One-sided tests

- For all $\beta_1 \leq \beta_{1,0}$,

$$\frac{\beta_1 - \beta_{1,0}}{\sqrt{\widehat{\text{Var}} [\hat{\beta}_1]}} \leq 0,$$

and

$$\begin{aligned} & \Pr \left[\frac{\hat{\beta}_1 - \beta_{1,0}}{\sqrt{\widehat{\text{Var}} [\hat{\beta}_1]}} > c_{1-\alpha} \mid \beta_1 \leq \beta_{1,0} \right] \\ &= \Pr \left[\frac{\hat{\beta}_1 - \beta_1}{\sqrt{\widehat{\text{Var}} [\hat{\beta}_1]}} + \frac{\beta_1 - \beta_{1,0}}{\sqrt{\widehat{\text{Var}} [\hat{\beta}_1]}} > c_{1-\alpha} \mid \beta_1 \leq \beta_{1,0} \right] \\ &\leq \Pr \left[\frac{\hat{\beta}_1 - \beta_1}{\sqrt{\widehat{\text{Var}} [\hat{\beta}_1]}} > c_{1-\alpha} \mid \beta_1 \leq \beta_{1,0} \right] \\ &= \alpha \text{ if } c_{1-\alpha} = t_{n-2, 1-\alpha}. \end{aligned}$$

One-sided tests

- For size α test, we reject $H_0 : \beta_1 \leq \beta_{1,0}$ against $H_1 : \beta_1 > \beta_{1,0}$ when

$$T = \frac{\hat{\beta}_1 - \beta_{1,0}}{\sqrt{\widehat{\text{Var}} [\hat{\beta}_1]}} > t_{n-2, 1-\alpha}.$$

where $t_{n-2, 1-\alpha}$ is the critical value corresponding to t -distribution with $n - 2$ degrees of freedom.

- Note that we use $1 - \alpha$ and not $1 - \alpha/2$ for choosing critical values in the case of one-sided testing.
- For size α test, we reject $H_0 : \beta_1 \geq \beta_{1,0}$ against $H_1 : \beta_1 < \beta_{1,0}$ when

$$T = \frac{\hat{\beta}_1 - \beta_{1,0}}{\sqrt{\widehat{\text{Var}} [\hat{\beta}_1]}} < -t_{n-2, 1-\alpha}.$$

One-sided tests

- One-sided p -values for $H_0 : \beta_1 \leq \beta_{1,0}$ against $H_1 : \beta_1 > \beta_{1,0}$:
 1. Compute T .
 2. Find τ such that $T = t_{n-2, 1-\tau}$.
 3. The p -value $= \tau$.