Supplement to "Generic Covariate Adjustment for Regression Discontinuity Designs"[*]

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This supplement contains the proofs of Theorems 1–5 of the main text. Section, equation, and lemma numbers in this supplement carry the prefix "S". Numbered references without this prefix, such as Theorem 1, Assumption 2, and (5), refer to the main text.

Notation. $\max_i$ ($\min_i$) is understood as $\max_{1\leq i\leq n}$ ($\min_{1\leq i\leq n}$). For real numbers a and b, $a\wedge b:=\min\{a,b\}$. For a vector x and a matrix A, $x^{[j]}$ denotes the j-th coordinate of x and $\mathrm{A}^{[jk]}$ denotes the jk-th element of A. For a square matrix A, $\|\mathrm{A}\|$ is understood as the spectral norm of A and $\mathrm{mineig}(\mathrm{A})$ denotes its smallest eigenvalue. "$a\lesssim b$" is understood as $a\leq C\cdot b$, for some universal constant $C>0$ that does not depend on the distribution of the variables or the sample size but may depend on the kernel function. Denote $K_{+,p}(u):=K(u)\,r_p(u)\,\mathbb{1}(u>0)$ and let $K_{-,p}$ be defined analogously. Let H be the $(p+1)\times(p+1)$ diagonal matrix with $(1,h,\ldots,h^p)$ on the diagonal. We adopt the following abbreviations: "wpa1" for "with probability approaching one", "LIE" for "law of iterated expectations", and "DCT" for "dominated convergence theorem". In the proofs, for notational brevity, we suppress the dependence on the bandwidth h: we write $\Pi_{+,p}$, $W_{+,p,i}$, $V_{p,i}$, and $\widehat{\lambda}_p$ for $\Pi_{+,p}(h)$, $W_{+,p,i}(h)$, $V_{p,i}(h)$, and $\widehat{\lambda}_p(h)$, and we drop h from the arguments of the estimators, writing $\widehat{\vartheta}_p^{eb}$ for $\widehat{\vartheta}_p^{eb}(h)$, and similarly for related quantities.

S.1 Proof of Theorem 1

Denote

$$\bar{W}_{+,p,i}:=\mathrm{e}_{p+1,1}^{\top}\left(\varphi\Lambda_{+,p}\right)^{-1}K_{+,p}\left(\frac{X_i}{h}\right),$$

let $\bar{W}_{-,p,i}$ be defined analogously, and let $\bar{W}_{p,i}:=\bar{W}_{+,p,i}-\bar{W}_{-,p,i}$. It follows from Chebyshev's inequality, a change of variables, a Taylor expansion, and the continuity of f_X that $\Pi_{+,p}-\mathrm{E}\left[\Pi_{+,p}\right]=O_p\left((nh)^{-1/2}\right)$. By a change of variables and the continuity of f_X, $\mathrm{E}\left[\Pi_{+,p}\right]=\varphi\Lambda_{+,p}+o(1)$. It now follows that

$$\left|\mathrm{mineig}\left(\Pi_{+,p}\right)-\mathrm{mineig}\left(\varphi\Lambda_{+,p}\right)\right|\leq\left\|\Pi_{+,p}-\varphi\Lambda_{+,p}\right\|=o_p(1)\,. \tag{S.1}$$

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Then, it follows from this result, the equality

$$\mathrm{A}^{-1}-\mathrm{B}^{-1}=-\mathrm{B}^{-1}(\mathrm{A}-\mathrm{B})\,\mathrm{B}^{-1}+\mathrm{B}^{-1}(\mathrm{A}-\mathrm{B})\,\mathrm{A}^{-1}(\mathrm{A}-\mathrm{B})\,\mathrm{B}^{-1} \tag{S.2}$$

for positive definite matrices A and B, and $\operatorname{mineig}(\Lambda_{+,p})>0$ that

$$\left\|\Pi_{+,p}^{-1}-(\varphi\Lambda_{+,p})^{-1}\right\| = o_p(1)\,. \tag{S.3}$$

Similar results hold for $\Pi_{-,p}$.

Lemma S.1. *Let A denote a random variable, and let $\{(A_i,X_i)\}_{i=1}^n$ be i.i.d. copies of (A,X). Suppose that Assumptions 1(ii) and 2 hold. Assume that $nh\uparrow\infty$ and $h\downarrow 0$. Let $\mathbb{B}\subseteq[\underline{x},\overline{x}]$ denote an open neighborhood of 0. The following results hold for all $k\in\mathbb{N}$. (i) If μ_A is uniformly continuous on $\mathbb{B}\setminus\{0\}$, then*

$$\mathrm{E}\left[\frac{1}{h}\bar{W}_{+,p}^{k}A\right]=\frac{\mu_{A,+}\omega_{+,p,0}^{0,k}}{\varphi^{k-1}}+o(1)\,.$$

(ii) If μ_A is $(p+1)$-times continuously differentiable with uniformly continuous $\mu_A^{(p+1)}$ on $\mathbb{B}\setminus\{0\}$, then

$$\frac{1}{nh}\sum_i W_{+,p,i}\mu_A(X_i)=\mu_{A,+}+\frac{\mu_{A,+}^{(p+1)}}{(p+1)!}\omega_{+,p,0}^{p+1,1}h^{p+1}+o_p\left(h^{p+1}\right).$$

(iii) If for some $\delta>0$, $\mu_{|A|^{1+\delta}}$ is bounded on $\mathbb{B}\setminus\{0\}$, then

$$\frac{1}{nh}\sum_i W_{+,p,i}^{k}A_i-\mathrm{E}\left[\frac{1}{h}\bar{W}_{+,p}^{k}A\right]=o_p(1)\,.$$

(iv) If the conditional density $f_{A|X}(\cdot\mid x)$ of A given $X=x$ with respect to some dominating measure m is dominated by $g(\cdot)$, uniformly in $x\in\mathbb{B}$, with $\int|y|^r g(y)\,m(\mathrm{d}y)<\infty$ for some $r>0$, then we have $\max_i|W_{+,p,i}A_i|=o_p\left((nh)^{1/r}\right)$. Similar results with "+" replaced by "−" also hold.

Proof of Lemma S.1. For Part (i), note that

$$\begin{aligned}
\mathrm{E}\left[\frac{1}{h}\bar{W}_{+,p}^{k}A\right] &= \int_0^{\overline{x}}\frac{1}{h}\left(\mathrm{e}_{p+1,1}^{\top}(\varphi\Lambda_{+,p})^{-1}r_p\left(\frac{x}{h}\right)\right)^k K^k\left(\frac{x}{h}\right)\mu_A(x)\,f_X(x)\,\mathrm{d}x\\
&= \int_0^{\frac{\overline{x}}{h}}\left(\mathrm{e}_{p+1,1}^{\top}(\varphi\Lambda_{+,p})^{-1}r_p(y)\right)^k K^k(y)\,\mu_A(hy)\,f_X(hy)\,\mathrm{d}y\\
&= \frac{\mu_{A,+}\omega_{+,p,0}^{0,k}}{\varphi^{k-1}}+o(1)
\end{aligned}$$

where the first equality follows from the LIE, the second equality follows from a change of variables, and the third equality follows from the continuity of μ_A and f_X, and the DCT.

By Taylor's theorem, for $X_i > 0$,

$$\mu_A(X_i) = \mu_+^\top r_p(X_i) + \frac{\mu_A^{(p+1)}(\dot{t}X_i)}{(p+1)!} X_i^{p+1}, \tag{S.4}$$

for some $\dot{t} \in (0,1)$, where $\mu_+ \coloneqq \left(\mu_{A,+}, \mu_{A,+}^{(1)}/1!, \ldots, \mu_{A,+}^{(p)}/p!\right)^\top$. Then, we write

$$\frac{1}{nh}\sum_i W_{+,p,i}\mu_A(X_i) = \frac{1}{nh}\sum_i W_{+,p,i}\left(r_p^\top(X_i)\mu_+\right) + \frac{1}{nh}\sum_i W_{+,p,i}\frac{\mu_A^{(p+1)}(\dot{t}X_i)}{(p+1)!}X_i^{p+1}. \tag{S.5}$$

By the definition of $W_{+,p,i}$, the first term on the right-hand side of (S.5) equals $\mu_{A,+}$. Write

$$\begin{aligned}\frac{1}{nh}\sum_i W_{+,p,i}\frac{\mu_A^{(p+1)}(\dot{t}X_i)}{(p+1)!}X_i^{p+1} &= \frac{1}{nh}\sum_i W_{+,p,i}\frac{\left(\mu_A^{(p+1)}(\dot{t}X_i) - \mu_{A,+}^{(p+1)}\right)}{(p+1)!}X_i^{p+1} \\ &\quad + \frac{\mu_{A,+}^{(p+1)}}{(p+1)!}\left(\frac{1}{nh}\sum_i W_{+,p,i}X_i^{p+1}\right).\end{aligned} \tag{S.6}$$

For the first term on the right-hand side, we have

$$\begin{aligned}&\left|\frac{1}{nh}\sum_i W_{+,p,i}\frac{\left(\mu_A^{(p+1)}(\dot{t}X_i) - \mu_{A,+}^{(p+1)}\right)}{(p+1)!}X_i^{p+1}\right| \\ \leq\ &\left(\frac{1}{nh}\sum_i\left|W_{+,p,i}\frac{(X_i/h)^{p+1}}{(p+1)!}\right|\right)\left(\sup_{0<x<h}\left|\mu_A^{(p+1)}(x) - \mu_{A,+}^{(p+1)}\right|\right)h^{p+1}.\end{aligned}$$

By Markov's inequality and (S.3), $(nh)^{-1}\sum_i\left|W_{+,p,i}(X_i/h)^{p+1}\right| = O_p(1)$ and therefore, the first term on the right-hand side of (S.6) is $o_p\left(h^{p+1}\right)$. It now follows from this result, (S.5), and (S.6) that

$$\frac{1}{nh}\sum_i W_{+,p,i}\mu_A(X_i) = \mu_{A,+} + \frac{\mu_{A,+}^{(p+1)}}{(p+1)!}\left(\frac{1}{nh}\sum_i W_{+,p,i}X_i^{p+1}\right) + o_p\left(h^{p+1}\right). \tag{S.7}$$

By (S.3), Chebyshev's inequality and a change of variables,

$$\frac{1}{nh}\sum_i W_{+,p,i}\left(\frac{X_i}{h}\right)^{p+1} = \omega_{+,p,0}^{p+1,1} + o_p(1). \tag{S.8}$$

Therefore, the second term on the right-hand side of (S.7) equals $\left(\mu_{A,+}^{(p+1)}\omega_{+,p,0}^{p+1,1}h^{p+1}\right)/(p+1)! + o_p\left(h^{p+1}\right)$. Part (ii) follows from this result and (S.7).

For Part (iii), write

$$\frac{1}{nh}\sum_i W^k_{+,p,i}A_i - \mathrm{E}\left[\frac{1}{h}\bar{W}^k_{+,p}A\right] = \frac{1}{nh}\sum_i \left(W^k_{+,p,i} - \bar{W}^k_{+,p,i}\right)A_i + \left\{\frac{1}{nh}\sum_i \bar{W}^k_{+,p,i}A_i - \mathrm{E}\left[\frac{1}{h}\bar{W}^k_{+,p}A\right]\right\}. \tag{S.9}$$

Then, by the binomial theorem, the boundedness of $\mu_{|A|}(\cdot)$, Markov's inequality, and (S.3), the first term on the right-hand side of (S.9) is $o_p(1)$. By the von Bahr–Esseen inequality (see, e.g., Lin and Bai, 2010, inequality 9.3.a) and the elementary inequality $\mathrm{E}\left[|A-\mathrm{E}[A]|^{1+\delta}\right] \le 2^{1+\delta}\mathrm{E}\left[|A|^{1+\delta}\right]$,

$$\begin{aligned}\mathrm{E}\left[\left|\frac{1}{nh}\sum_i \bar{W}^k_{+,p,i}A_i - \mathrm{E}\left[\frac{1}{h}\bar{W}^k_{+,p}A\right]\right|^{1+\delta}\right] &\lesssim \frac{1}{(nh)^{1+\delta}}\sum_i \mathrm{E}\left[\left|\bar{W}^k_{+,p,i}A_i\right|^{1+\delta}\right] \\ &= o(1),\end{aligned} \tag{S.10}$$

where the equality follows from (S.3) and the boundedness of $\mu_{|A|^{1+\delta}}(\cdot)$. Now Markov's inequality yields that the second term on the right-hand side of (S.9) is $o_p(1)$. Part (iii) follows from these results.

For Part (iv), first we note that $\max_i |W_{+,p,i}A_i| \le \left\|\Pi^{-1}_{+,p}\right\| \max_i \|K_{+,p}(X_i/h)\| |A_i|$. It follows from (S.3) that $\left\|\Pi^{-1}_{+,p}\right\| = O_p(1)$. Fix an arbitrary $\varepsilon > 0$. By the union bound,

$$\begin{aligned}&\Pr\left[(nh)^{-1/r}\max_i \left\|K_{+,p}\left(\frac{X_i}{h}\right)\right\| |A_i| > \varepsilon\right] \\ \le\ & \sum_i \Pr\left[\left\|K_{+,p}\left(\frac{X_i}{h}\right)\right\|^r |A_i|^r > (nh)\,\varepsilon^r\right] \\ \le\ & \mathrm{E}\left[\frac{1}{h}\left\|K_{+,p}\left(\frac{X_i}{h}\right)\right\|^r |A|^r \mathbb{1}\left(|A| > \frac{(nh)^{1/r}\varepsilon}{\max_{u\in\mathbb{R}}\|K_{+,p}(u)\|}\right)\right]/\varepsilon^r.\end{aligned} \tag{S.11}$$

Since $nh \uparrow \infty$, by the DCT, the right-hand side of the second inequality above converges to zero as $n \uparrow \infty$. It now follows that $\max_i \|K_{+,p}(X_i/h)\| |A_i| = o_p\left((nh)^{1/r}\right)$. ■

By the monotonicity of $x \mapsto \left(x^{1+\varrho}-1\right)/(\varrho(1+\varrho))$ on $(0,\infty)$, $\widehat{\lambda}_p$ maximizes

$$P_\varrho(\lambda) := \frac{1}{\varrho}\sum_i \left(1 + V^\top_{p,i}\lambda\right)^{\varrho/(1+\varrho)}$$

subject to $1 + V^\top_{p,i}\lambda > 0$ for all $i = 1,\ldots,n$. It follows from simple calculations that $P_\varrho(\cdot)$ is concave on the constraint set.

Lemma S.2. *Suppose that the assumptions in the statement of Theorem 1 hold. Then,* $\widehat{\lambda}_p =$

$O_p\left((nh)^{-1/2}\right)$ *and*

$$\widehat{\lambda}_p = (1+\varrho)\left(\sum_i V_{p,i}V_{p,i}^\top\right)^{-1}\sum_i V_{p,i} + o_p\left((nh)^{-1/2}\right). \tag{S.12}$$

Proof. Denote $\mathbb{L} \coloneqq \left\{\lambda : \|\lambda\| \leq (nh)^{-1/(2+\delta)}\right\}$. By Lemma S.1(iv), we have $\max_i \sup_{\lambda\in\mathbb{L}} \left|V_{p,i}^\top\lambda\right| = o_p(1)$. The set $\mathbb{L}$ is contained in the constraint set $\left\{\lambda : 1+V_{p,i}^\top\lambda > 0,\, \forall i\right\}$ wpa1, and $\bar{\lambda}_p \coloneqq \arg\max_{\lambda\in\mathbb{L}} P_\varrho(\lambda)$ exists wpa1. By a Taylor expansion,

$$\begin{aligned} P_\varrho(0) &\leq P_\varrho\left(\bar{\lambda}_p\right) \\ &= P_\varrho(0) + \frac{1}{1+\varrho}\left(\sum_i V_{p,i}\right)^\top \bar{\lambda}_p - \frac{1}{2(1+\varrho)^2}\sum_i \frac{\left(V_{p,i}^\top\bar{\lambda}_p\right)^2}{\left(1+\dot{t}\cdot V_{p,i}^\top\bar{\lambda}_p\right)^{1/(1+\varrho)+1}}, \end{aligned} \tag{S.13}$$

for some $\dot{t} \in (0,1)$. By the continuity of $x \mapsto x^{-1/(1+\varrho)-1}$ at $x=1$ and $\max_i \left|V_{p,i}^\top\bar{\lambda}_p\right| = o_p(1)$, we have $\min_i \left(1+\dot{t}\cdot V_{p,i}^\top\bar{\lambda}_p\right)^{-1/(1+\varrho)-1} > 1/2$ wpa1. It now follows that $\operatorname{mineig}\left(\sum_i V_{p,i}V_{p,i}^\top\right)\|\bar{\lambda}_p\| \leq 4|1+\varrho|\,\|\sum_i V_{p,i}\|$ wpa1. By Lemma S.1(i) and (iii),

$$\operatorname{mineig}\left(\frac{1}{nh}\sum_i V_{p,i}V_{p,i}^\top\right) = \left(\frac{\omega_{p,0}^{0,2}}{\varphi}\right)\operatorname{mineig}\left(\mu_{\bar{Z}\bar{Z}^\top,\pm}\right) + o_p(1)$$

and under our assumptions, $\mu_{\bar{Z}\bar{Z}^\top,\pm}$ is positive definite and $\operatorname{mineig}\left(\mu_{\bar{Z}\bar{Z}^\top,\pm}\right) > 0$.

Let $\xi_i \coloneqq Z_i - \mu_Z(X_i)$. By Lemma S.1(ii),

$$\begin{aligned} \frac{1}{nh}\sum_i W_{p,i}Z_i &= \frac{1}{nh}\sum_i W_{p,i}\mu_Z(X_i) + \frac{1}{nh}\sum_i W_{p,i}\xi_i \\ &= \frac{1}{nh}\sum_i W_{p,i}\xi_i + \frac{\left(\mu_{Z,+}^{(p+1)}\omega_{+,p,0}^{p+1,1} - \mu_{Z,-}^{(p+1)}\omega_{-,p,0}^{p+1,1}\right)}{(p+1)!}h^{p+1} + o_p\left(h^{p+1}\right). \end{aligned} \tag{S.14}$$

By Chebyshev's inequality and a change of variables, we have $(nh)^{-1/2}\sum_i K_{s,p}(X_i/h)\xi_i = O_p(1)$ for $s \in \{-,+\}$, and therefore, by this result and (S.3),

$$\begin{aligned} \frac{1}{\sqrt{nh}}\sum_i W_{p,i}\xi_i^\top &= \mathrm{e}_{p+1,1}^\top\Pi_{+,p}^{-1}\left(\frac{1}{\sqrt{nh}}\sum_i K_{+,p}\left(\frac{X_i}{h}\right)\xi_i^\top\right) - \mathrm{e}_{p+1,1}^\top\Pi_{-,p}^{-1}\left(\frac{1}{\sqrt{nh}}\sum_i K_{-,p}\left(\frac{X_i}{h}\right)\xi_i^\top\right) \\ &= O_p(1). \end{aligned} \tag{S.15}$$

By this result and (S.14), we have $(nh)^{-1/2}\sum_i W_{p,i}Z_i = O_p(1)$. Then, it follows from this result and

$$\frac{1}{\sqrt{nh}}\sum_i V_{p,i} = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{nh}}\sum_i W_{p,i}Z_i \end{pmatrix}, \tag{S.16}$$

that

$$\frac{1}{\sqrt{nh}}\sum_{i}V_{p,i}=O_{p}\left(1\right). \tag{S.17}$$

Combining the inequality $\operatorname{mineig}\left(\sum_i V_{p,i}V_{p,i}^{\top}\right)\left\|\bar{\lambda}_p\right\| \leq 4\left|1+\varrho\right|\left\|\sum_i V_{p,i}\right\|$ with the established fact that $\operatorname{mineig}\left(\left(nh\right)^{-1}\sum_i V_{p,i}V_{p,i}^{\top}\right)$ is bounded away from 0 wpa1 and (S.17) yields $\left\|\bar{\lambda}_p\right\| = O_p\left(\left(nh\right)^{-1/2}\right)$. Since $\left(nh\right)^{-1/2}/\left(nh\right)^{-1/(2+\delta)} \to 0$, $\bar{\lambda}_p$ lies in the interior of $\mathbb{L}$ wpa1. The objective P_ϱ is concave on the open convex constraint set $\left\{\lambda : 1+V_{p,i}^{\top}\lambda > 0,\, \forall i\right\}$, which contains $\mathbb{L}$ wpa1. Hence any local maximizer of P_ϱ in the interior of $\mathbb{L}$ is also its global maximizer over the full constraint set, and so $\widehat{\lambda}_p = \bar{\lambda}_p$ wpa1. In particular, $\widehat{\lambda}_p = O_p\left(\left(nh\right)^{-1/2}\right)$, which establishes the first claim of the lemma. Since $\widehat{\lambda}_p \in \mathbb{L}$ wpa1, the bound $\max_i \sup_{\lambda\in\mathbb{L}}\left|V_{p,i}^{\top}\lambda\right| = o_p\left(1\right)$ noted above implies $\max_i\left|V_{p,i}^{\top}\widehat{\lambda}_p\right| = o_p\left(1\right)$.

Denote $\iota_\varrho\left(x\right) \coloneqq x^{-1/(1+\varrho)}$. The first-order condition is given by $\sum_i V_{p,i}\iota_\varrho\left(1+V_{p,i}^{\top}\widehat{\lambda}_p\right)=0$. By a Taylor expansion,

$$\sum_{i}V_{p,i}\left\{\iota_{\varrho}\left(1\right)+\iota_{\varrho}^{\prime}\left(1\right)\left(V_{p,i}^{\top}\widehat{\lambda}_{p}\right)+\frac{1}{2}\iota_{\varrho}^{\prime\prime}\left(1+\dot{t}\cdot V_{p,i}^{\top}\widehat{\lambda}_{p}\right)\left(V_{p,i}^{\top}\widehat{\lambda}_{p}\right)^{2}\right\}=0, \tag{S.18}$$

for some $\dot{t}\in(0,1)$. Using $\iota_\varrho\left(1\right)=1$ and $\iota_\varrho^{\prime}\left(1\right)=-1/\left(1+\varrho\right)$, (S.18) rearranges to

$$\widehat{\lambda}_{p}=\left(1+\varrho\right)\left(\sum_{i}V_{p,i}V_{p,i}^{\top}\right)^{-1}\sum_{i}V_{p,i}+\frac{1+\varrho}{2}\left(\sum_{i}V_{p,i}V_{p,i}^{\top}\right)^{-1}\sum_{i}\iota_{\varrho}^{\prime\prime}\left(1+\dot{t}\cdot V_{p,i}^{\top}\widehat{\lambda}_{p}\right)\left(V_{p,i}^{\top}\widehat{\lambda}_{p}\right)^{2}V_{p,i}. \tag{S.19}$$

Since $\widehat{\lambda}_p\in\mathbb{L}$ wpa1 gives $\max_i\left|V_{p,i}^{\top}\widehat{\lambda}_p\right| = o_p\left(1\right)$, by the continuity of $\iota_\varrho^{\prime\prime}$ at 1,

$$\max_{i}\left|\iota_{\varrho}^{\prime\prime}\left(1+\dot{t}\cdot V_{p,i}^{\top}\widehat{\lambda}_{p}\right)\right|\leq\sup_{|t|\leq\max_{i}\left|V_{p,i}^{\top}\widehat{\lambda}_{p}\right|}\left|\iota_{\varrho}^{\prime\prime}\left(1+t\right)\right|=O_{p}\left(1\right). \tag{S.20}$$

Hence, by this result, the triangle inequality, and $\left\|\left(nh\right)^{-1}\sum_i V_{p,i}V_{p,i}^{\top}\right\| = O_p\left(1\right)$,

$$\begin{aligned}\left\|\frac{1}{nh}\sum_{i}\iota_{\varrho}^{\prime\prime}\left(1+\dot{t}\cdot V_{p,i}^{\top}\widehat{\lambda}_{p}\right)\left(V_{p,i}^{\top}\widehat{\lambda}_{p}\right)^{2}V_{p,i}\right\| &\leq O_{p}\left(1\right)\cdot\max_{i}\left\|V_{p,i}\right\|\cdot\left\|\frac{1}{nh}\sum_{i}V_{p,i}V_{p,i}^{\top}\right\|\left\|\widehat{\lambda}_{p}\right\|^{2}\\ &= o_{p}\left(\left(nh\right)^{-1/2}\right).\end{aligned}$$

It follows that the remainder term on the right-hand side of (S.19) is $o_p\left(\left(nh\right)^{-1/2}\right)$. Substituting back into (S.19) yields the second claim of the lemma. ■

By a Taylor expansion similar to that in (S.18), applied to the scalar $\iota_\varrho\left(1+V_{p,i}^\top\widehat{\lambda}_p\right)$, we have

$$\iota_\varrho\left(1+V_{p,i}^\top\widehat{\lambda}_p\right)=1-\frac{V_{p,i}^\top\widehat{\lambda}_p}{1+\varrho}+\frac{1}{2}\iota_\varrho''\left(1+\dot{t}\cdot V_{p,i}^\top\widehat{\lambda}_p\right)\left(V_{p,i}^\top\widehat{\lambda}_p\right)^2, \tag{S.21}$$

for some $\dot{t}\in(0,1)$. Averaging (S.21) over $i=1,\dots,n$ and bounding the two remainder terms, we have $n^{-1}\sum_i V_{p,i}^\top\widehat{\lambda}_p=O_p\left(n^{-1}\right)$ by (S.17) and Lemma S.2, and also

$$\left|\frac{1}{n}\sum_i\iota_\varrho''\left(1+\dot{t}\cdot V_{p,i}^\top\widehat{\lambda}_p\right)\left(V_{p,i}^\top\widehat{\lambda}_p\right)^2\right|\leq O_p(1)\left\|\widehat{\lambda}_p\right\|^2\left\|\frac{1}{n}\sum_i V_{p,i}V_{p,i}^\top\right\|=O_p\left(n^{-1}\right)$$

by Lemma S.1(i, iii) and Lemma S.2. Hence $n^{-1}\sum_i\iota_\varrho(1+V_{p,i}^\top\widehat{\lambda}_p)=1+O_p\left(n^{-1}\right)$, and substituting (S.21) into (5) yields

$$\widehat{w}_{p,i}=\frac{1}{n}\left\{\iota_\varrho(1)+\iota_\varrho'(1)\left(V_{p,i}^\top\widehat{\lambda}_p\right)+\frac{1}{2}\iota_\varrho''\left(1+\dot{t}\cdot V_{p,i}^\top\widehat{\lambda}_p\right)\left(V_{p,i}^\top\widehat{\lambda}_p\right)^2\right\}\left\{1+O_p\left(n^{-1}\right)\right\}, \tag{S.22}$$

where $\dot{t}\in(0,1)$ denotes the mean value.

Let

$$\Pi_{+,p}^{eb}:=\frac{1}{h}\sum_i\widehat{w}_{p,i}K\left(\frac{X_i}{h}\right)I_i r_p\left(\frac{X_i}{h}\right)r_p^\top\left(\frac{X_i}{h}\right)\text{ and }W_{+,p,i}^{eb}:=\mathrm{e}_{p+1,1}^\top\left(\Pi_{+,p}^{eb}\right)^{-1}K_{+,p}\left(\frac{X_i}{h}\right),$$

and let $\left(\Pi_{-,p}^{eb},W_{-,p,i}^{eb}\right)$ be defined similarly. Let $W_{p,i}^{eb}:=W_{+,p,i}^{eb}-W_{-,p,i}^{eb}$.

It follows from (S.22) and Lemma S.2 that $\left\|\Pi_{+,p}^{eb}-\Pi_{+,p}\right\|=O_p\left((nh)^{-1/2}\right)$. By the same arguments used to show (S.3),

$$\left\|\left(\Pi_{+,p}^{eb}\right)^{-1}-\Pi_{+,p}^{-1}\right\|=O_p\left((nh)^{-1/2}\right). \tag{S.23}$$

A similar result holds for $\Pi_{-,p}^{eb}$.

Proof of Theorem 1. It is clear that $\widehat{\vartheta}_p^{eb}$ has a closed form given by

$$\widehat{\vartheta}_p^{eb}=\frac{1}{h}\sum_i\widehat{w}_{p,i}W_{p,i}^{eb}Y_i. \tag{S.24}$$

Denote $G_{p,i}:=W_{p,i}^{eb}Y_i$. Since by (S.3) and (S.23), $\left\|\left(\Pi_{s,p}^{eb}\right)^{-1}\right\|=O_p(1)$ and $\left\|\Pi_{s,p}^{-1}\right\|=O_p(1)$, $(nh)^{-1}\sum_i|G_{p,i}|\left\|V_{p,i}\right\|=O_p(1)$ follows easily from these results and Markov's inequality. Then, by this result, (S.22), $\widehat{\lambda}_p = O_p\left((nh)^{-1/2}\right)$ from Lemma S.2 and $\max_i \left|V_{p,i}^\top \widehat{\lambda}_p\right| = o_p(1)$, we have

$$\widehat{\vartheta}_p^{eb} = \frac{1}{nh}\sum_i G_{p,i}\left\{\iota_\varrho(1) + \iota_\varrho'(1)\left(V_{p,i}^\top \widehat{\lambda}_p\right)\right\} + o_p\left((nh)^{-1/2}\right).$$

Then, by (S.12) and (S.17),

$$\widehat{\vartheta}_p^{eb} = \frac{1}{nh}\sum_i G_{p,i} - \left(\frac{1}{nh}\sum_i G_{p,i}V_{p,i}^\top\right)\left(\frac{1}{nh}\sum_i V_{p,i}V_{p,i}^\top\right)^{-1}\frac{1}{nh}\sum_i V_{p,i} + o_p\left((nh)^{-1/2}\right). \quad \text{(S.25)}$$

By (S.2), $\left\|\Pi_{+,p}^{eb} - \Pi_{+,p}\right\| = O_p\left((nh)^{-1/2}\right)$, and (S.23),

$$\frac{1}{nh}\sum_i\left(W_{+,p,i}^{eb} - W_{+,p,i}\right)Y_i = -\mathrm{e}_{p+1,1}^\top \Pi_{+,p}^{-1}\left(\Pi_{+,p}^{eb} - \Pi_{+,p}\right)\widehat{\beta}_+ + o_p\left((nh)^{-1/2}\right), \quad \text{(S.26)}$$

where $\widehat{\beta}_+ \coloneqq (nh)^{-1}\sum_i \Pi_{+,p}^{-1}K_{+,p}(X_i/h)\,Y_i$. By (S.22), (S.26), and the definition in (2), we have

$$\frac{1}{nh}\sum_i\left(W_{+,p,i}^{eb} - W_{+,p,i}\right)Y_i = \frac{1}{1+\varrho}\left(\frac{1}{nh}\sum_i W_{+,p,i}^2\left(\widehat{\lambda}_p^\top \bar{Z}_i\right)\left(r_p^\top\left(\frac{X_i}{h}\right)\widehat{\beta}_+\right)\right) + o_p\left((nh)^{-1/2}\right).$$

By standard arguments, $\widehat{\beta}_+ = \mu_{Y,+}\mathrm{e}_{p+1,1} + o_p(1)$. By this result and Lemma S.1,

$$\begin{aligned}\frac{1}{nh}\sum_i W_{+,p,i}^2\bar{Z}_i\left(r_p^\top\left(\frac{X_i}{h}\right)\widehat{\beta}_+\right) &= \left(\frac{\omega_{p,0}^{0,2}}{\varphi}\right)\mu_{\bar{Z}}\mu_{Y,+} + o_p(1)\\ \frac{1}{nh}\sum_i V_{p,i}V_{p,i}^\top &= \left(\frac{\omega_{p,0}^{0,2}}{\varphi}\right)\mu_{\bar{Z}\bar{Z}^\top,\pm} + o_p(1),\end{aligned} \quad \text{(S.27)}$$

where $\mu_{\bar{Z}}$ denotes the common value of $\mu_{\bar{Z},+}$ and $\mu_{\bar{Z},-}$, which exists under Assumption 1(i). It now follows from these results and Lemma S.2 that

$$\frac{1}{nh}\sum_i\left(W_{+,p,i}^{eb} - W_{+,p,i}\right)Y_i = \left(\frac{1}{nh}\sum_i V_{p,i}\right)^\top \mu_{\bar{Z}\bar{Z}^\top,\pm}^{-1}\mu_{\bar{Z}}\mu_{Y,+} + o_p\left((nh)^{-1/2}\right).$$

Similarly,

$$\frac{1}{nh}\sum_i\left(W_{-,p,i}^{eb} - W_{-,p,i}\right)Y_i = -\left(\frac{1}{nh}\sum_i V_{p,i}\right)^\top \mu_{\bar{Z}\bar{Z}^\top,\pm}^{-1}\mu_{\bar{Z}}\mu_{Y,-} + o_p\left((nh)^{-1/2}\right).$$

Recall that $\bar{Z} \coloneqq \begin{pmatrix} 1 & Z^\top \end{pmatrix}^\top$. Clearly, $\mu_{\bar{Z}\bar{Z}^\top,\pm}\mathrm{e}_{d_z+1,1} = 2\mu_{\bar{Z}}$. Then, by these results and (S.16), we have

$$\begin{aligned}\frac{1}{nh}\sum_i G_{p,i}-\frac{1}{nh}\sum_i W_{p,i}Y_i &= \left(\frac{1}{nh}\sum_i V_{p,i}\right)^{\top} \mathrm{e}_{d_z+1,1}\left(\frac{\mu_{Y,\pm}}{2}\right)+o_p\left((nh)^{-1/2}\right)\\ &= o_p\left((nh)^{-1/2}\right).\end{aligned} \tag{S.28}$$

Next, by Lemma S.1 and (S.23),

$$\frac{1}{nh}\sum_i G_{p,i}V_{p,i}^{\top}=\left(\frac{\omega_{p,0}^{0,2}}{\varphi}\right)\mu_{Y\bar{Z}^{\top},\pm}+o_p\left(1\right). \tag{S.29}$$

Therefore, by this result and (S.27),

$$\left(\frac{1}{nh}\sum_i G_{p,i}V_{p,i}^{\top}\right)\left(\frac{1}{nh}\sum_i V_{p,i}V_{p,i}^{\top}\right)^{-1}=\mu_{Y\bar{Z}^{\top},\pm}\mu_{\bar{Z}\bar{Z}^{\top},\pm}^{-1}+o_p\left(1\right). \tag{S.30}$$

We write $\mu_{Y\bar{Z}^{\top},\pm}=\left[\begin{array}{cc}\mu_{Y,\pm} & \mu_{YZ^{\top},\pm}\end{array}\right]$, write $\mu_{\bar{Z}\bar{Z}^{\top},\pm}$ as a block matrix and apply the block inversion formula to get

$$\left[\begin{array}{cc}2 & \mu_{Z^{\top},\pm}\\ \mu_{Z,\pm} & \mu_{ZZ^{\top},\pm}\end{array}\right]^{-1}=\left[\begin{array}{cc}\frac{1}{2}+\frac{1}{4}\mu_{Z^{\top},\pm}\mathrm{S}^{-1}\mu_{Z,\pm} & -\frac{1}{2}\mu_{Z^{\top},\pm}\mathrm{S}^{-1}\\ -\frac{1}{2}\mathrm{S}^{-1}\mu_{Z,\pm} & \mathrm{S}^{-1}\end{array}\right], \tag{S.31}$$

where $\mathrm{S}\coloneqq\mu_{ZZ^{\top},\pm}-\mu_{Z,\pm}\mu_{Z^{\top},\pm}/2=\mathrm{Var}_{\pm}\left[Z\right]$. By this result and (S.16), we get

$$\mu_{Y\bar{Z}^{\top},\pm}\mu_{\bar{Z}\bar{Z}^{\top},\pm}^{-1}\left(\frac{1}{nh}\sum_i V_{p,i}\right)=\frac{1}{nh}\sum_i W_{p,i}Z_i^{\top}\gamma.$$

Then by this result and (S.30), we have

$$\left(\frac{1}{nh}\sum_i G_{p,i}V_{p,i}^{\top}\right)\left(\frac{1}{nh}\sum_i V_{p,i}V_{p,i}^{\top}\right)^{-1}\frac{1}{nh}\sum_i V_{p,i}=\frac{1}{nh}\sum_i W_{p,i}Z_i^{\top}\gamma+o_p\left((nh)^{-1/2}\right).$$

Note that this result, (S.25) and (S.28) yield that $\widehat{\vartheta}_p^{eb}=(nh)^{-1}\sum_i W_{p,i}\epsilon_i+o_p\left((nh)^{-1/2}\right)$, where $\epsilon_i\coloneqq Y_i-Z_i^{\top}\gamma$. Note that $\widehat{\vartheta}_p^{CCFT}$ has the same asymptotic representation under Assumption 1(i). The conclusion follows from these observations. ■

S.2 Proof of Theorem 2

Let

$$\widehat{\kappa}_{+,p}\left(\tau\right)\coloneqq\arg\min_b\sum_i\widehat{w}_{p,i}K\left(\frac{X_i}{h}\right)I_i\rho_{\tau}\left(Y_i-r_p^{\top}\left(X_i\right)b\right)$$

and let $\widehat{\kappa}_{-,p}(\tau)$ be defined by the same formula with I_i replaced by $\mathbb{1}(X_i < 0)$. Clearly, $\widehat{\vartheta}_p^{eb}(\tau) = \mathrm{e}_{p+1,1}^{\top}(\widehat{\kappa}_{+,p}(\tau) - \widehat{\kappa}_{-,p}(\tau))$. Let $\psi_\tau(u) \coloneqq \tau - \mathbb{1}(u \le 0)$, $\kappa_+(\tau) \coloneqq \left(\kappa_{Y,+}(\tau), \kappa_{Y,+}^{(1)}(\tau), \ldots, \kappa_{Y,+}^{(p)}(\tau)/p!\right)^{\top}$ and

$$\begin{aligned} R_+(\varDelta, \tau) &\coloneqq \sqrt{\frac{n}{h}} \sum_i \widehat{w}_{p,i} K_{+,p}\left(\frac{X_i}{h}\right) \psi_\tau\left(\tilde{Y}_{+,i} - \frac{r_p^{\top}(X_i/h)\varDelta}{\sqrt{nh}}\right), \text{ where} \\ \tilde{Y}_{+,i} &\coloneqq Y_i - r_p^{\top}(X_i)\kappa_+(\tau). \end{aligned}$$

Let $\mathbb{P}g$ denote $\mathrm{E}[g(Y,X,Z)]$ and let $\mathbb{P}_n g$ denote the sample average. Let $\mathbb{G}_n g \coloneqq \sqrt{n}(\mathbb{P}_n - \mathbb{P})g$ and let $\{\mathbb{G}_n g : g \in \mathscr{G}\}$ denote the empirical process indexed by $g \in \mathscr{G}$, where $\mathscr{G}$ is a function class. Let $\|\mathbb{G}_n\|_{\mathscr{G}} \coloneqq \sup_{g \in \mathscr{G}} |\mathbb{G}_n g|$. The proof of Theorem 2 follows an adaptation of the approach in Koenker and Zhao (1994).

Lemma S.3. *Suppose that the assumptions in the statement of Theorem 2 hold. For all $M > 0$,*

$$\sup_{\|\varDelta\| \le M, \tau \in [\underline{\tau}, \overline{\tau}]} \|R_+(\varDelta, \tau) + D_+(\tau)\varDelta - R_+(0,\tau)\| = o_p(1), \tag{S.32}$$

where $D_+(\tau) \coloneqq \varphi_+(\tau)\varphi\Lambda_{+,p}$.

Proof of Lemma S.3. By (S.22), we have

$$\begin{aligned} & R_+(\varDelta,\tau) - R_+(0,\tau) \\ = \; & \sqrt{\frac{n}{h}} \sum_i \widehat{w}_{p,i} K_{+,p}\left(\frac{X_i}{h}\right)\left\{\mathbb{1}\left(\tilde{Y}_{+,i} \le 0\right) - \mathbb{1}\left(\tilde{Y}_{+,i} \le \frac{r_p^{\top}(X_i/h)\varDelta}{\sqrt{nh}}\right)\right\} \\ = \; & T_1(\varDelta,\tau)\left\{1 + O_p\left(n^{-1}\right)\right\} + T_2(\varDelta,\tau)\left\{1 + O_p\left(n^{-1}\right)\right\}, \end{aligned}$$

where

$$\begin{aligned} T_1(\varDelta,\tau) &\coloneqq \frac{1}{\sqrt{nh}} \sum_i K_{+,p}\left(\frac{X_i}{h}\right)\left\{\mathbb{1}\left(\tilde{Y}_{+,i} \le 0\right) - \mathbb{1}\left(\tilde{Y}_{+,i} \le \frac{r_p^{\top}(X_i/h)\varDelta}{\sqrt{nh}}\right)\right\} \\ T_2(\varDelta,\tau) &\coloneqq \frac{1}{\sqrt{nh}} \sum_i K_{+,p}\left(\frac{X_i}{h}\right)\left(\iota_\varrho\left(1 + V_{p,i}^{\top}\widehat{\lambda}_p\right) - 1\right)\left\{\mathbb{1}\left(\tilde{Y}_{+,i} \le 0\right) - \mathbb{1}\left(\tilde{Y}_{+,i} \le \frac{r_p^{\top}(X_i/h)\varDelta}{\sqrt{nh}}\right)\right\}. \end{aligned}$$

We decompose

$$\begin{aligned} T_1(\varDelta,\tau) &= \{T_1(\varDelta,\tau) - \mathrm{E}[T_1(\varDelta,\tau)]\} + \mathrm{E}[T_1(\varDelta,\tau)] \\ &= \frac{1}{\sqrt{n}} \sum_i \left\{\frac{1}{\sqrt{h}} K_{+,p}\left(\frac{X_i}{h}\right)\left(\mathbb{1}\left(\tilde{Y}_{+,i} \le 0\right) - \mathbb{1}\left(\tilde{Y}_{+,i} \le \frac{r_p^{\top}(X_i/h)\varDelta}{\sqrt{nh}}\right)\right)\right. \\ &\qquad \left. - \mathrm{E}\left[\frac{1}{\sqrt{h}} K_{+,p}\left(\frac{X}{h}\right)\left(\mathbb{1}\left(\tilde{Y}_+ \le 0\right) - \mathbb{1}\left(\tilde{Y}_+ \le \frac{r_p^{\top}(X/h)\varDelta}{\sqrt{nh}}\right)\right)\right]\right\} \end{aligned}$$

$$+\sqrt{nh}\cdot \mathrm{E}\left[\frac{1}{h}K_{+,p}\left(\frac{X}{h}\right)\left(\mathbb{1}\left(\tilde{Y}_{+}\leq 0\right)-\mathbb{1}\left(\tilde{Y}_{+}\leq \frac{r_{p}^{\top}\left(X/h\right)\mathit{\Delta}}{\sqrt{nh}}\right)\right)\right]. \tag{S.33}$$

Under Assumption 4(i)–(iii), for $x = hu$ with $u \in (0,1]$, we have

$$\begin{aligned} & F_{Y|X}\left(r_{p}^{\top}\left(x\right)\kappa_{+}\left(\tau\right)\mid x\right)-F_{Y|X}\left(r_{p}^{\top}\left(x\right)\kappa_{+}\left(\tau\right)+\frac{r_{p}^{\top}\left(x/h\right)\mathit{\Delta}}{\sqrt{nh}}\mid x\right) \\ = \ & \left\{-f_{Y|X}\left(Q_{Y|X}\left(\tau\mid x\right)\mid x\right)+o\left(1\right)\right\}\frac{r_{p}^{\top}\left(x/h\right)\mathit{\Delta}}{\sqrt{nh}}, \end{aligned} \tag{S.34}$$

uniformly for all $\|\mathit{\Delta}\| \leq M, \tau \in [\underline{\tau}, \overline{\tau}]$. By this result and the LIE, we have

$$\begin{aligned} & \frac{1}{\sqrt{nh}}\mathrm{E}\left[T_{1}\left(\mathit{\Delta},\tau\right)\right] \\ = \ & \mathrm{E}\left[\frac{1}{h}K_{+,p}\left(\frac{X}{h}\right)\left(F_{Y|X}\left(r_{p}^{\top}\left(X\right)\kappa_{+}\left(\tau\right)\mid X\right)-F_{Y|X}\left(r_{p}^{\top}\left(X\right)\kappa_{+}\left(\tau\right)+\frac{r_{p}^{\top}\left(X/h\right)\mathit{\Delta}}{\sqrt{nh}}\mid X\right)\right)\right] \\ = \ & \left\{-\mathrm{E}\left[\frac{1}{h}K_{+,p}\left(\frac{X}{h}\right)r_{p}^{\top}\left(\frac{X}{h}\right)f_{Y|X}\left(Q_{Y|X}\left(\tau\mid X\right)\mid X\right)\right]+o\left(1\right)\right\}\frac{\mathit{\Delta}}{\sqrt{nh}} \\ = \ & -D_{+}\left(\tau\right)\frac{\mathit{\Delta}}{\sqrt{nh}}+o\left(\left(nh\right)^{-1/2}\right), \end{aligned}$$

uniformly for all $\|\mathit{\Delta}\| \leq M, \tau \in [\underline{\tau}, \overline{\tau}]$. For any $j = 1, \ldots, p+1$, write $T_1^{[j]}(\mathit{\Delta}, \tau) - \mathrm{E}\left[T_1^{[j]}(\mathit{\Delta}, \tau)\right] = \mathbb{G}_n g_1(\cdot \mid \mathit{\Delta}, \tau)$, where

$$g_{1}\left(y,x\mid \mathit{\Delta},\tau\right):=\frac{1}{\sqrt{h}}K_{+,p}^{[j]}\left(\frac{x}{h}\right)\left\{\mathbb{1}\left(y\leq r_{p}^{\top}\left(x\right)\kappa_{+}\left(\tau\right)\right)-\mathbb{1}\left(y\leq r_{p}^{\top}\left(x\right)\left(\kappa_{+}\left(\tau\right)+\frac{\mathrm{H}^{-1}\mathit{\Delta}}{\sqrt{nh}}\right)\right)\right\}. \tag{S.35}$$

Let $\mathscr{G}_1 := \{g_1(\cdot \mid \mathit{\Delta}, \tau) : \|\mathit{\Delta}\| \leq M,\ \tau \in [\underline{\tau}, \overline{\tau}]\}$. By Markov's inequality, it suffices to show that $\mathrm{E}\left[\|\mathbb{G}_n\|_{\mathscr{G}_1}\right] = o(1)$. The function class

$$\left\{(y,x)\mapsto \mathbb{1}\left(y\leq r_{p}^{\top}\left(x\right)b\right): b\in \mathbb{R}^{p+1}\right\} \tag{S.36}$$

is Vapnik–Chervonenkis (VC)-subgraph with index no more than $p+4$, because its elements are indicators of nonnegativity sets of functions in a $(p+2)$-dimensional vector space (Kosorok, 2007, Lemma 9.6). Since multiplying by a fixed function preserves the VC property (Kosorok, 2007, Lemma 9.9(vi)), by Kosorok (2007, Theorem 9.3) and Chernozhukov et al. (2014, Lemma A.6), $\mathscr{G}_1$ is VC-type (see, e.g., Chen and Kato, 2020, Definition 2.1) with a constant envelope $G_1 \lesssim h^{-1/2}$ and characteristics independent of n.

Note that for all a, b,

$$\left(\mathbb{1}\left(x\leq a\right)-\mathbb{1}\left(x\leq b\right)\right)^{2}=\left|\mathbb{1}\left(x\leq a\right)-\mathbb{1}\left(x\leq b\right)\right|=\mathbb{1}\left(\min\left\{a,b\right\}<x\leq \max\left\{a,b\right\}\right). \tag{S.37}$$

By this result, the LIE, (S.34), and a change of variables,

$$
\begin{aligned}
& \mathbb{P} g_1^2(\cdot \mid \Delta, \tau) \\
\leq\ & \mathrm{E}\left[\frac{1}{h}\left\|K_{+,p}\left(\frac{X}{h}\right)\right\|^2\left|F_{Y|X}\left(r_p^{\top}(X)\,\kappa_{+}(\tau) \mid X\right)-F_{Y|X}\left(r_p^{\top}(X)\,\kappa_{+}(\tau)+\frac{r_p^{\top}(X/h)\,\Delta}{\sqrt{nh}} \mid X\right)\right|\right] \\
=\ & O\left((nh)^{-1/2}\right),
\end{aligned} \tag{S.38}
$$

uniformly in $\|\Delta\| \leq M, \tau \in [\underline{\tau}, \overline{\tau}]$. By Chen and Kato (2020, Corollary 5.5),

$$
\begin{aligned}
\mathrm{E}\left[\|\mathbb{G}_n\|_{\mathscr{G}_1}\right] & \lesssim \sqrt{\left(\sup_{\|\Delta\|\leq M,\, \tau\in[\underline{\tau},\overline{\tau}]} \mathbb{P} g_1^2(\cdot \mid \Delta, \tau)\right)\log(n)}+\frac{\log(n)}{\sqrt{nh}} \\
& = O\left(\frac{\sqrt{\log(n)}}{(nh)^{1/4}}+\frac{\log(n)}{\sqrt{nh}}\right).
\end{aligned} \tag{S.39}
$$

Under $nh \to \infty$ and $(\log n)/\sqrt{nh} \to 0$, both terms are $o(1)$, so $\mathrm{E}\left[\|\mathbb{G}_n\|_{\mathscr{G}_1}\right] = o(1)$ holds and Markov's inequality yields $T_1(\Delta, \tau) - \mathrm{E}[T_1(\Delta, \tau)] = o_p(1)$ uniformly for all $\|\Delta\| \leq M, \tau \in [\underline{\tau}, \overline{\tau}]$. It now follows that

$$
T_1(\Delta, \tau) = -D_{+}(\tau)\,\Delta + o_p(1), \tag{S.40}
$$

and $T_1(\Delta, \tau) = O_p(1)$, uniformly for all $\|\Delta\| \leq M, \tau \in [\underline{\tau}, \overline{\tau}]$.

By the triangle inequality and (S.37), we have

$$
\begin{aligned}
& \sup_{\|\Delta\|\leq M} \|T_2(\Delta, \tau)\| \leq \\
& \quad \frac{1}{\sqrt{nh}}\sum_i \left\|K_{+,p}\left(\frac{X_i}{h}\right)\right\|\left|\iota_\varrho\left(1+V_{p,i}^{\top}\widehat{\lambda}_p\right)-1\right| \mathbb{1}\left(-\frac{\|r_p(X_i/h)\|\,M}{\sqrt{nh}} < \tilde{Y}_{+,i} \leq \frac{\|r_p(X_i/h)\|\,M}{\sqrt{nh}}\right).
\end{aligned}
$$

Since $\max_i \left|\iota_\varrho\left(1+V_{p,i}^{\top}\widehat{\lambda}_p\right)-1\right|$ does not depend on τ, the right-hand side is bounded by

$$
\max_i \left|\iota_\varrho\left(1+V_{p,i}^{\top}\widehat{\lambda}_p\right)-1\right| \left(\sqrt{n}\cdot\mathbb{P}_n g_2(\cdot \mid \tau)\right),
$$

where

$$
g_2(y, x \mid \tau) := \frac{1}{\sqrt{h}}\left\|K_{+,p}\left(\frac{x}{h}\right)\right\| \mathbb{1}\left(-\frac{\|r_p(x/h)\|\,M}{\sqrt{nh}} < y - r_p^{\top}(x)\,\kappa_{+}(\tau) \leq \frac{\|r_p(x/h)\|\,M}{\sqrt{nh}}\right).
$$

For the first factor, by Lemma S.2, $\widehat{\lambda}_p = O_p\left((nh)^{-1/2}\right)$, and $\max_i \|V_{p,i}\| = O_p\left((nh)^{1/(2+\delta)}\right)$, so $\max_i \left|V_{p,i}^{\top}\widehat{\lambda}_p\right| = o_p(1)$. Since ι_ϱ is continuously differentiable at 1 with $\iota_\varrho(1) = 1$, we have

$$
\max_i \left|\iota_\varrho\left(1+V_{p,i}^{\top}\widehat{\lambda}_p\right)-1\right| \leq \sup_{|t|\leq \max_i\left|V_{p,i}^{\top}\widehat{\lambda}_p\right|} \left|\iota_\varrho'(1+t)\right| \cdot \max_i \left|V_{p,i}^{\top}\widehat{\lambda}_p\right| = o_p(1).
$$

Since $\sqrt{n}\cdot\mathbb{P}_n g_2(\cdot\mid\tau)=\sqrt{n}\cdot\mathbb{P}g_2(\cdot\mid\tau)+\mathbb{G}_n g_2(\cdot\mid\tau)$, it suffices to bound $\sup_{\tau\in[\underline{\tau},\overline{\tau}]}\sqrt{n}\cdot\mathbb{P}g_2(\cdot\mid\tau)$. By a result similar to (S.34), we have $\sqrt{n}\cdot\mathbb{P}g_2(\cdot\mid\tau)=O(1)$, uniformly in τ. For the fluctuation term, note that by similar arguments, the class $\mathscr{G}_2:=\{g_2(\cdot\mid\tau):\tau\in[\underline{\tau},\overline{\tau}]\}$ is VC-type with characteristics independent of n and a constant envelope $G_2\lesssim h^{-1/2}$. By similar calculations, we can show that $\sup_{\tau\in[\underline{\tau},\overline{\tau}]}\mathbb{P}g_2^2(\cdot\mid\tau)$ is of order $(nh)^{-1/2}$. The same maximal inequality used in the proof of $\mathrm{E}\left[\|\mathbb{G}_n\|_{\mathscr{G}_1}\right]=o(1)$ yields $\mathrm{E}\left[\|\mathbb{G}_n\|_{\mathscr{G}_2}\right]=o(1)$. Therefore $T_2(\Delta,\tau)=o_p(1)$, uniformly for all $\|\Delta\|\le M,\tau\in[\underline{\tau},\overline{\tau}]$. ■

Let $\tilde{K}_{+,p}(u):=K^2(u)\,\mathbb{1}(u>0)\,r_p(u)\left(\mathrm{e}_{p+1,1}^\top\Lambda_{+,p}^{-1}r_p(u)\right)$ and let $\pi_+:=\int_0^1\tilde{K}_{+,p}(u)\,\mathrm{d}u$. Recall that $\xi_i:=Z_i-\mu_Z(X_i)$ and $U_i(\tau):=\tau-\mathbb{1}\left(Y_i\le Q_{Y\mid X}(\tau\mid X_i)\right)$. Let $\gamma_+(\tau):=\left(\mathrm{Var}_\pm[Z]\right)^{-1}\mu_{ZU(\tau),+}$ and let $\gamma_-(\tau)$ be defined similarly, so that $\gamma(\tau)=\gamma_+(\tau)/\varphi_+(\tau)+\gamma_-(\tau)/\varphi_-(\tau)$ by the definition of $\gamma(\tau)$ in Section 4.2.

Lemma S.4. *Suppose that the assumptions in the statement of Theorem 2 hold. We have the following linearization for $R_+(0,\tau)$:*

$$\begin{aligned}R_+(0,\tau)&=\frac{1}{\sqrt{nh}}\sum_i K_{+,p}\left(\frac{X_i}{h}\right)U_i(\tau)-\left(\frac{\pi_+\varphi}{\omega_{p,0}^{0,2}}\right)\gamma_+^\top(\tau)\left\{\frac{1}{\sqrt{nh}}\sum_i W_{p,i}\xi_i\right\}\\&\quad+\bar{\mathscr{B}}_+(\tau)\sqrt{nh}h^{p+1}+o_p(1),\end{aligned}$$

uniformly in $\tau\in[\underline{\tau},\overline{\tau}]$, where

$$\bar{\mathscr{B}}_+(\tau):=\left(\frac{\int_0^1 K_{+,p}(u)\,u^{p+1}\mathrm{d}u}{(p+1)!}\right)\varphi_+(\tau)\,\varphi\kappa_{Y,+}^{(p+1)}(\tau)-\left(\frac{\pi_+\varphi}{\omega_{p,0}^{0,2}}\right)\left\{\frac{\gamma_+^\top(\tau)\left(\mu_{Z,+}^{(p+1)}\omega_{+,p,0}^{p+1,1}-\mu_{Z,-}^{(p+1)}\omega_{-,p,0}^{p+1,1}\right)}{(p+1)!}\right\}.$$

Proof of Lemma S.4. We have

$$\begin{aligned}R_+(0,\tau)&=\sqrt{\frac{n}{h}}\sum_i\widehat{w}_{p,i}K_{+,p}\left(\frac{X_i}{h}\right)\psi_\tau\left(\tilde{Y}_{+,i}\right)\\&=T_1(\tau)\left(1+O_p\left(n^{-1}\right)\right)+T_2(\tau)\left(1+O_p\left(n^{-1}\right)\right),\end{aligned}$$

where

$$\begin{aligned}T_1(\tau)&:=\frac{1}{\sqrt{nh}}\sum_i K_{+,p}\left(\frac{X_i}{h}\right)\left\{1-\frac{V_{p,i}^\top\widehat{\lambda}_p}{1+\varrho}\right\}\psi_\tau\left(\tilde{Y}_{+,i}\right)\\T_2(\tau)&:=\frac{1}{\sqrt{nh}}\sum_i K_{+,p}\left(\frac{X_i}{h}\right)\left\{\iota_\varrho\left(1+V_{p,i}^\top\widehat{\lambda}_p\right)-\left(1-\frac{V_{p,i}^\top\widehat{\lambda}_p}{1+\varrho}\right)\right\}\psi_\tau\left(\tilde{Y}_{+,i}\right).\end{aligned}$$

By the Taylor expansion of ι_ϱ in (S.21),

$$\iota_\varrho\left(1+V_{p,i}^\top\widehat{\lambda}_p\right)-\left(1-\frac{V_{p,i}^\top\widehat{\lambda}_p}{1+\varrho}\right)=\frac{1}{2}\iota_\varrho''\left(1+\dot{t}\cdot V_{p,i}^\top\widehat{\lambda}_p\right)\left(V_{p,i}^\top\widehat{\lambda}_p\right)^2.$$

Using $|\psi_\tau(\cdot)| \le 1$, (S.20), $\max_i \left|V_{p,i}^\top \widehat{\lambda}_p\right| = o_p(1)$, $\left(V_{p,i}^\top \widehat{\lambda}_p\right)^2 \le \left|V_{p,i}^\top \widehat{\lambda}_p\right| \cdot \|V_{p,i}\| \cdot \left\|\widehat{\lambda}_p\right\|$, and

$$\frac{1}{nh}\sum_i \left\|K_{+,p}\left(\frac{X_i}{h}\right)\right\| \|V_{p,i}\| = O_p(1)$$

by Markov's inequality and $\left\|(nh)^{-1}\sum_i V_{p,i}V_{p,i}^\top\right\| = O_p(1)$, we have

$$\begin{aligned}\|T_2(\tau)\| &\lesssim \max_i \left|\iota_\varrho''\left(1+\dot{t}\cdot V_{p,i}^\top \widehat{\lambda}_p\right)\right| \cdot \max_i \left|V_{p,i}^\top \widehat{\lambda}_p\right| \cdot \left\|\widehat{\lambda}_p\right\| \cdot \frac{1}{\sqrt{nh}}\sum_i \left\|K_{+,p}\left(\frac{X_i}{h}\right)\right\| \|V_{p,i}\| \\ &= o_p(1).\end{aligned}$$

The bound is uniform in τ because no term on the right-hand side depends on τ.

Then,

$$T_1(\tau) = \frac{1}{\sqrt{nh}}\sum_i K_{+,p}\left(\frac{X_i}{h}\right)\psi_\tau\left(\tilde{Y}_{+,i}\right) - \frac{1}{1+\varrho}\cdot\frac{1}{\sqrt{nh}}\sum_i K_{+,p}\left(\frac{X_i}{h}\right)\psi_\tau\left(\tilde{Y}_{+,i}\right)V_{p,i}^\top \widehat{\lambda}_p. \tag{S.41}$$

We have

$$\frac{1}{\sqrt{nh}}\sum_i K_{+,p}\left(\frac{X_i}{h}\right)\psi_\tau\left(\tilde{Y}_{+,i}\right) = \frac{1}{\sqrt{nh}}\sum_i K_{+,p}\left(\frac{X_i}{h}\right)U_i(\tau) + T_3(\tau)$$

where

$$T_3(\tau) \coloneqq \frac{1}{\sqrt{nh}}\sum_i K_{+,p}\left(\frac{X_i}{h}\right)\left\{\mathbb{1}\left(Y_i \le Q_{Y|X}(\tau \mid X_i)\right) - \mathbb{1}\left(Y_i \le r_p^\top(X_i)\,\kappa_+(\tau)\right)\right\}. \tag{S.42}$$

Since $\mathrm{E}[U(\tau) \mid X] = 0$, the sum is centered: for any $j = 1, \ldots, p+1$,

$$\frac{1}{\sqrt{nh}}\sum_i K_{+,p}^{[j]}\left(\frac{X_i}{h}\right)U_i(\tau) = \mathbb{G}_n g_3(\cdot \mid \tau),$$

where

$$g_3(y, x \mid \tau) \coloneqq \frac{1}{\sqrt{h}}K_{+,p}^{[j]}\left(\frac{x}{h}\right)\left\{\tau - \mathbb{1}\left(y \le Q_{Y|X}(\tau \mid x)\right)\right\}.$$

By the monotonicity of $\tau \mapsto Q_{Y|X}(\tau \mid x)$, the function class

$$\left\{(y,x) \mapsto \mathbb{1}\left(y \le Q_{Y|X}(\tau \mid x)\right) : \tau \in [\underline{\tau}, \overline{\tau}]\right\} \tag{S.43}$$

is VC-subgraph with index 2 by Kosorok (2007, Lemma 9.10). Then it follows from Kosorok (2007, Theorem 9.3 and Lemma 9.9(vi)) and Chernozhukov et al. (2014, Lemma A.6) that the class $\mathscr{G}_3 \coloneqq \{g_3(\cdot \mid \tau) : \tau \in [\underline{\tau}, \overline{\tau}]\}$ is VC-type with characteristics independent of n and an envelope $G_3(x) \coloneqq h^{-1/2}\|K_{+,p}(x/h)\|$. Therefore, the uniform entropy integral is finite and independent of n (see, e.g., calculations in the proof of Chernozhukov et al., 2014, Corollary 5.1). By a change of variables, $\mathbb{P}G_3^2 = O(1)$.

By the maximal inequality in van der Vaart and Wellner (1996, Theorem 2.14.1),

$$\mathrm{E}\left[\|\mathbb{G}_n\|_{\mathscr{G}_3}\right] \lesssim \sqrt{\mathbb{P}G_3^2} = O(1). \tag{S.44}$$

Markov's inequality yields that $\mathbb{G}_n g_3(\cdot \mid \tau) = O_p(1)$, uniformly in $\tau \in [\underline{\tau}, \overline{\tau}]$. For the bias term $T_3(\tau)$, we write $T_3(\tau) = T_4(\tau) + T_5(\tau)$, where

$$\begin{aligned}
T_4(\tau) &:= \frac{1}{\sqrt{nh}} \sum_i K_{+,p}\left(\frac{X_i}{h}\right)\Big\{\Big(\mathbb{1}\left(Y_i \le Q_{Y|X}(\tau \mid X_i)\right) - \mathbb{1}\left(Y_i \le r_p^\top(X_i)\,\kappa_+(\tau)\right)\Big) \\
&\quad - \left(\tau - F_{Y|X}\left(r_p^\top(X_i)\,\kappa_+(\tau) \mid X_i\right)\right)\Big\} \\
T_5(\tau) &:= \frac{1}{\sqrt{nh}} \sum_i K_{+,p}\left(\frac{X_i}{h}\right)\left\{\tau - F_{Y|X}\left(r_p^\top(X_i)\,\kappa_+(\tau) \mid X_i\right)\right\}.
\end{aligned}$$

Under Assumption 4(ii) and (iii), as $x \downarrow 0$,

$$F_{Y|X}\left(Q_{Y|X}(\tau \mid x) \mid x\right) - F_{Y|X}\left(r_p^\top(x)\,\kappa_+(\tau) \mid x\right) = \frac{x^{p+1}}{(p+1)!}\left(\varphi_+(\tau)\,\kappa_{Y,+}^{(p+1)}(\tau) + o(1)\right), \tag{S.45}$$

uniformly in $\tau \in [\underline{\tau}, \overline{\tau}]$.

For any $j = 1, \ldots, p+1$, write the summand of $T_4^{[j]}(\tau)$ as $g_4(Y_i, X_i \mid \tau) - g_5(X_i \mid \tau)$, where

$$\begin{aligned}
g_4(y, x \mid \tau) &:= \frac{1}{\sqrt{h}} K_{+,p}^{[j]}\left(\frac{x}{h}\right)\left\{\mathbb{1}\left(y \le Q_{Y|X}(\tau \mid x)\right) - \mathbb{1}\left(y \le r_p^\top(x)\,\kappa_+(\tau)\right)\right\}, \\
g_5(x \mid \tau) &:= \frac{1}{\sqrt{h}} K_{+,p}^{[j]}\left(\frac{x}{h}\right)\left\{\tau - F_{Y|X}\left(r_p^\top(x)\,\kappa_+(\tau) \mid x\right)\right\}.
\end{aligned}$$

By the LIE, $\mathrm{E}\left[g_4(Y, X \mid \tau) \mid X\right] = g_5(X \mid \tau)$, so $\mathbb{P}g_4(\cdot \mid \tau) = \mathbb{P}g_5(\cdot \mid \tau)$ and $T_4(\tau) = \mathbb{G}_n(g_4 - g_5)(\cdot \mid \tau)$. Define $\mathscr{G}_4 := \{(g_4 - g_5)(\cdot \mid \tau) : \tau \in [\underline{\tau}, \overline{\tau}]\}$. We have shown above that both (S.36) and (S.43) are VC-subgraph. Since the subgraph inequality $t \le F_{Y|X}\left(r_p^\top(x)\,b \mid x\right)$ for $t \in (0,1]$ is equivalent to $r_p^\top(x)\,b \ge Q_{Y|X}(t \mid x)$ (van der Vaart, 1998, Lemma 21.1(i)), the subgraph of $x \mapsto F_{Y|X}\left(r_p^\top(x)\,b \mid x\right)$ equals

$$\left\{(x,t) \in \mathbb{R} \times (0,1] : r_p^\top(x)\,b - Q_{Y|X}(t \mid x) \ge 0\right\} \cup \left(\mathbb{R} \times (-\infty, 0]\right).$$

The former set can be represented as a non-negativity set of the $(p+2)$-dimensional vector space spanned by $r_p(\cdot)$ and $Q_{Y|X}(\cdot \mid \cdot)$. Therefore, by Kosorok (2007, Lemma 9.6) and preservation results in Kosorok (2007, Lemma 9.9), $\left\{(y,x) \mapsto F_{Y|X}\left(r_p^\top(x)\,b \mid x\right) : b \in \mathbb{R}^{p+1}\right\}$ is VC-subgraph with index at most $p+4$. It now follows from Kosorok (2007, Theorem 9.3 and Lemma 9.9(vi)) and Chernozhukov et al. (2014, Lemma A.6) that $\mathscr{G}_4$ is VC-type with characteristics independent of n and a constant envelope $G_4 \lesssim h^{-1/2}$. Since $\mathrm{E}\left[\left(g_4(Y, X \mid \tau) - g_5(X \mid \tau)\right)^2 \mid X\right] \le \mathrm{E}\left[g_4^2(Y, X \mid \tau) \mid X\right]$, by the LIE, (S.45) and (S.37), we have

$$\begin{aligned}\mathbb{P}\left(g_4-g_5\right)^2(\cdot\mid\tau) &\leq \mathbb{P}g_4^2(\cdot\mid\tau)\\ &= \mathrm{E}\left[\frac{1}{h}\left(K_{+,p}^{[j]}\right)^2\left(\frac{X}{h}\right)\left|F_{Y|X}\left(Q_{Y|X}(\tau\mid X)\mid X\right)-F_{Y|X}\left(r_p^\top(X)\kappa_+(\tau)\mid X\right)\right|\right]\\ &= O\left(h^{p+1}\right), \qquad \text{(S.46)}\end{aligned}$$

uniformly in $\tau\in[\underline{\tau},\overline{\tau}]$. By Chen and Kato (2020, Corollary 5.5),

$$\mathrm{E}\left[\sup_{\tau\in[\underline{\tau},\overline{\tau}]}\left|\mathbb{G}_n\left(g_4-g_5\right)(\cdot\mid\tau)\right|\right]=O\left(\sqrt{h^{p+1}\log(n)}+\frac{\log(n)}{\sqrt{nh}}\right).$$

Markov's inequality yields that $T_4(\tau)=o_p(1)$, uniformly in $\tau\in[\underline{\tau},\overline{\tau}]$.

We can write $T_5^{[j]}(\tau)-\mathrm{E}\left[T_5^{[j]}(\tau)\right]=\mathbb{G}_n g_5(\cdot\mid\tau)$. By similar arguments, the function class $\mathscr{G}_5:=\{g_5(\cdot\mid\tau):\tau\in[\underline{\tau},\overline{\tau}]\}$ is VC-type with characteristics independent of n and admits an envelope given by

$$G_5(x):=\frac{1}{\sqrt{h}}\left|K_{+,p}^{[j]}\left(\frac{x}{h}\right)\right|\sup_{\tau\in[\underline{\tau},\overline{\tau}]}\left|\tau-F_{Y|X}\left(r_p^\top(x)\kappa_+(\tau)\mid x\right)\right|.$$

By a change of variables and (S.45), $\mathbb{P}G_5^2=O\left(h^{2(p+1)}\right)$. It follows from the same arguments used to prove (S.44) that $\mathrm{E}\left[\|\mathbb{G}_n\|_{\mathscr{G}_5}\right]=O\left(h^{p+1}\right)$. Markov's inequality yields that $\mathbb{G}_n g_5(\cdot\mid\tau)=o_p(1)$, uniformly in $\tau\in[\underline{\tau},\overline{\tau}]$. Therefore, $T_5(\tau)-\mathrm{E}[T_5(\tau)]=o_p(1)$, uniformly in $\tau\in[\underline{\tau},\overline{\tau}]$.

By (S.45), we have

$$\begin{aligned}\frac{1}{\sqrt{nh}}\mathrm{E}[T_5(\tau)] &= \mathrm{E}\left[\frac{1}{h}K_{+,p}\left(\frac{X}{h}\right)\left\{F_{Y|X}\left(Q_{Y|X}(\tau\mid X)\mid X\right)-F_{Y|X}\left(r_p^\top(X)\kappa_+(\tau)\mid X\right)\right\}\right]\\ &= h^{p+1}\left\{\left(\frac{\int_0^1 K_{+,p}(u)\,u^{p+1}\mathrm{d}u}{(p+1)!}\right)\varphi_+(\tau)\,\kappa_{Y,+}^{(p+1)}(\tau)\,\varphi+o(1)\right\},\end{aligned}$$

uniformly in $\tau\in[\underline{\tau},\overline{\tau}]$. Combining these results, we have

$$\begin{aligned}\frac{1}{\sqrt{nh}}\sum_i K_{+,p}\left(\frac{X_i}{h}\right)\psi_\tau\left(\tilde{Y}_{+,i}\right) &= \sqrt{nh}h^{p+1}\left\{\left(\frac{\int_0^1 K_{+,p}(u)\,u^{p+1}\mathrm{d}u}{(p+1)!}\right)\varphi_+(\tau)\,\kappa_{Y,+}^{(p+1)}(\tau)\,\varphi+o(1)\right\}\\ &\quad+\frac{1}{\sqrt{nh}}\sum_i K_{+,p}\left(\frac{X_i}{h}\right)U_i(\tau)+o_p(1), \qquad \text{(S.47)}\end{aligned}$$

uniformly in $\tau\in[\underline{\tau},\overline{\tau}]$.

Next, we derive the linearization for the second term on the right-hand side of (S.41). Note that for all $\tau\in[\underline{\tau},\overline{\tau}]$,

$$\left\|\frac{1}{nh}\sum_i K_{+,p}\left(\frac{X_i}{h}\right)\psi_\tau\left(\tilde{Y}_{+,i}\right)V_{p,i}^\top-\left(\frac{1}{nh}\sum_i\tilde{K}_{+,p}\left(\frac{X_i}{h}\right)\psi_\tau\left(\tilde{Y}_{+,i}\right)\bar{Z}_i^\top\right)/\varphi\right\|$$

$$
\begin{aligned}
&\leq \left(\frac{2}{nh}\sum_i K^2\left(\frac{X_i}{h}\right)\left\|r_p\left(\frac{X_i}{h}\right)\right\|^2\|\bar{Z}_i\|\right)\left(\left\|\Pi_{+,p}^{-1}-\varphi^{-1}\Lambda_{+,p}^{-1}\right\|\right)\\
&= o_p(1). \qquad \text{(S.48)}
\end{aligned}
$$

Write

$$
\frac{1}{nh}\sum_i \tilde{K}_{+,p}\left(\frac{X_i}{h}\right)\psi_\tau\left(\tilde{Y}_{+,i}\right)\bar{Z}_i^\top = T_6(\tau)+T_7(\tau),
$$

where

$$
\begin{aligned}
T_6(\tau) &:= \frac{1}{nh}\sum_i \tilde{K}_{+,p}\left(\frac{X_i}{h}\right)U_i(\tau)\bar{Z}_i^\top\\
T_7(\tau) &:= \frac{1}{nh}\sum_i \tilde{K}_{+,p}\left(\frac{X_i}{h}\right)\left\{\mathbb{1}\left(Y_i\leq Q_{Y|X}(\tau\mid X_i)\right)-\mathbb{1}\left(Y_i\leq r_p^\top(X_i)\kappa_+(\tau)\right)\right\}\bar{Z}_i^\top. \qquad \text{(S.49)}
\end{aligned}
$$

By Assumption 3(iv), Assumption 4(i), (v), and (vi), and the DCT, $(\tau,x)\mapsto \mu_{U(\tau)\bar{Z}^\top}(x)$ is jointly uniformly continuous on $[\underline{\tau},\overline{\tau}]\times\mathbb{B}_0$. By this result and the LIE, we have

$$
\mathrm{E}\left[T_6(\tau)\right] = \pi_+\varphi\mu_{U(\tau)\bar{Z}^\top,+}+o(1),
$$

uniformly in $\tau\in[\underline{\tau},\overline{\tau}]$. By Assumption 4(i) and (v), for all $z\in\operatorname{supp}(Z)$, as $x\downarrow 0$,

$$
\sup_{\tau\in[\underline{\tau},\overline{\tau}]}\left|F_{Y|XZ}\left(Q_{Y|X}(\tau\mid x)\mid x,z\right)-F_{Y|XZ}\left(r_p^\top(x)\kappa_+(\tau)\mid x,z\right)\right| = o(1).
$$

By this result, Assumption 4(vi) and the DCT, as $x\downarrow 0$,

$$
\sup_{\tau\in[\underline{\tau},\overline{\tau}]}\left|\mathrm{E}\left[\left\{F_{Y|XZ}\left(Q_{Y|X}(\tau\mid X)\mid X,Z\right)-F_{Y|XZ}\left(r_p^\top(X)\kappa_+(\tau)\mid X,Z\right)\right\}\bar{Z}\mid X=x\right]\right| = o(1).
$$

By this result and the LIE, we have

$$
\begin{aligned}
\mathrm{E}\left[T_7(\tau)\right] &= \mathrm{E}\left[\frac{1}{h}\tilde{K}_{+,p}\left(\frac{X}{h}\right)\left\{F_{Y|XZ}\left(Q_{Y|X}(\tau\mid X)\mid X,Z\right)-F_{Y|XZ}\left(r_p^\top(X)\kappa_+(\tau)\mid X,Z\right)\right\}\bar{Z}\right]\\
&= o(1),
\end{aligned}
$$

uniformly in $\tau\in[\underline{\tau},\overline{\tau}]$.

Working coordinate-wise, fix $j\in\{1,\ldots,p+1\}$ and $k\in\{1,\ldots,d_z+1\}$. We write

$$
T_6^{[jk]}(\tau)-\mathrm{E}\left[T_6^{[jk]}(\tau)\right] = \frac{1}{\sqrt{nh}}\mathbb{G}_n g_6(\cdot\mid\tau),
$$

where $\bar{z}:=\left(1,z^\top\right)^\top$ and

$$
g_6(y,x,z\mid\tau) := \frac{1}{\sqrt{h}}\tilde{K}_{+,p}^{[j]}\left(\frac{x}{h}\right)\left\{\tau-\mathbb{1}\left(y\leq Q_{Y|X}(\tau\mid x)\right)\right\}\bar{z}^{[k]}.
$$

By similar arguments, we can show that $\mathscr{G}_6 := \{g_6(\cdot \mid \tau) : \tau \in [\underline{\tau}, \overline{\tau}]\}$ is VC-type with characteristics independent of n and admits an envelope $G_6(y, x, z) := h^{-1/2} \left\| \tilde{K}_{+,p}(x/h) \right\| \left| \bar{z}^{[k]} \right|$. By the LIE, Assumption 3(iv) (which gives $\mathrm{E}\left[\left(\bar{Z}^{[k]} \right)^2 \mid X = x \right]$ bounded uniformly in x near 0), and a change of variables, $\mathbb{P}G_6^2 = O(1)$. By the maximal inequality in van der Vaart and Wellner (1996, Theorem 2.14.1), $\mathrm{E}\left[\|\mathbb{G}_n\|_{\mathscr{G}_6} \right] = O(1)$. It follows that $T_6(\tau) - \mathrm{E}[T_6(\tau)] = o_p(1)$, uniformly in $\tau \in [\underline{\tau}, \overline{\tau}]$. It follows from the same arguments that $T_7(\tau) - \mathrm{E}[T_7(\tau)] = o_p(1)$, uniformly in $\tau \in [\underline{\tau}, \overline{\tau}]$.

Therefore,

$$\frac{1}{nh} \sum_i K_{+,p}\left(\frac{X_i}{h} \right) \psi_\tau \left(\tilde{Y}_{+,i} \right) V_{p,i}^\top = \pi_+ \mu_{U(\tau)\bar{Z}^\top,+} + o_p(1),$$

uniformly in $\tau \in [\underline{\tau}, \overline{\tau}]$. By this result, Lemma S.2 and $(nh)^{-1} \sum_i V_{p,i} V_{p,i}^\top = \left(\omega_{p,0}^{0,2} / \varphi \right) \mu_{\bar{Z}\bar{Z}^\top,\pm} + o_p(1)$,

$$\begin{aligned}
& \frac{1}{1+\varrho} \cdot \frac{1}{\sqrt{nh}} \sum_i K_{+,p}\left(\frac{X_i}{h} \right) \psi_\tau \left(\tilde{Y}_{+,i} \right) V_{p,i}^\top \widehat{\lambda}_p \\
= \quad & \pi_+ \mu_{U(\tau)\bar{Z}^\top,+} \left(\left(\frac{\omega_{p,0}^{0,2}}{\varphi} \right) \mu_{\bar{Z}\bar{Z}^\top,\pm} \right)^{-1} \frac{1}{\sqrt{nh}} \sum_i V_{p,i} + o_p(1) \\
= \quad & \left(\frac{\pi_+ \varphi}{\omega_{p,0}^{0,2}} \right) \gamma_+^\top(\tau) \left\{ \frac{1}{\sqrt{nh}} \sum_i W_{p,i} \xi_i + \frac{\left(\mu_{Z,+}^{(p+1)} \omega_{+,p,0}^{p+1,1} - \mu_{Z,-}^{(p+1)} \omega_{-,p,0}^{p+1,1} \right)}{(p+1)!} \sqrt{nh} h^{p+1} \right\} + o_p(1),
\end{aligned}$$

uniformly in $\tau \in [\underline{\tau}, \overline{\tau}]$. The conclusion follows from this result, (S.41), (S.47) and $R_+(0,\tau) = T_1(\tau) + o_p(1)$. ■

Let $\mathit{\Delta}_+(\tau) := \sqrt{nh} \mathrm{H}\left(\widehat{\kappa}_{+,p}(\tau) - \kappa_+(\tau) \right)$. Recall that H is the diagonal matrix with $(1, h, \ldots, h^p)$ on its diagonal. Let $\mathit{\Delta}_-(\tau)$ be defined analogously. Then we have

$$\sqrt{nh} \left(\widehat{\vartheta}_p^{eb}(\tau) - \vartheta(\tau) \right) = \mathrm{e}_{p+1,1}^\top \left(\mathit{\Delta}_+(\tau) - \mathit{\Delta}_-(\tau) \right). \tag{S.50}$$

The following lemma provides an asymptotic linearization for $\mathit{\Delta}_+(\tau)$. A similar result holds for $\mathit{\Delta}_-(\tau)$.

Lemma S.5. *Suppose that the assumptions in the statement of Theorem 2 hold. We have the following linearization:*

$$\mathit{\Delta}_+(\tau) = D_+^{-1}(\tau) R_+(0,\tau) + o_p(1),$$

uniformly in $\tau \in [\underline{\tau}, \overline{\tau}]$.

Proof of Lemma S.5. By Cai and Xu (2008, Lemma A.2), we have

$$\left\| \sum_i \widehat{w}_{p,i} K_{+,p}\left(\frac{X_i}{h} \right) \psi_\tau \left(Y_i - r_p^\top(X_i) \widehat{\kappa}_{+,p}(\tau) \right) \right\| \leq (p+1) \max_i \left\| \widehat{w}_{p,i} K_{+,p}\left(\frac{X_i}{h} \right) \right\|, \tag{S.51}$$

where the left-hand side equals $\sqrt{h/n}\,\|R_+(\varDelta_+(\tau),\tau)\|$ by the definitions of $R_+(\varDelta,\tau)$ and $\varDelta_+(\tau)$. We can show that the right-hand side is $O_p(n^{-1})$. Indeed, by (S.22), $\max_i \left|V_{p,i}^\top\widehat{\lambda}_p\right| = o_p(1)$, (S.20) and $\iota_\varrho(1)=1$, we have $\max_i n\widehat{w}_{p,i} = 1+o_p(1)$. Hence,

$$(p+1)\max_i \left\|\widehat{w}_{p,i}K_{+,p}\left(\frac{X_i}{h}\right)\right\| \lesssim \max_i \widehat{w}_{p,i} = O_p\left(n^{-1}\right).$$

These results imply that $R_+(\varDelta_+(\tau),\tau) = O_p((nh)^{-1/2})$, uniformly in $\tau\in[\underline{\tau},\overline{\tau}]$.

Since

$$-\varDelta^\top R_+(\lambda\varDelta,\tau) = \sqrt{\frac{n}{h}}\sum_i \widehat{w}_{p,i}K\left(\frac{X_i}{h}\right)\mathbb{1}(X_i>0)\left(-\varDelta^\top r_p\left(\frac{X_i}{h}\right)\right)\psi_\tau\left(\tilde{Y}_{+,i}-\frac{\lambda\left(r_p^\top(X_i/h)\,\varDelta\right)}{\sqrt{nh}}\right)$$

and $\psi_\tau(\cdot)$ is increasing, we have

$$-\varDelta^\top R_+(\lambda\varDelta,\tau) \geq -\varDelta^\top R_+(\varDelta,\tau), \tag{S.52}$$

for all $\lambda\geq 1$.

For all $M>0$, $\eta>0$, by the union bound, we have

$$\begin{aligned}
&\Pr\left[\inf_{\|\varDelta\|=M,\,\tau\in[\underline{\tau},\overline{\tau}]} -\varDelta^\top R_+(\varDelta,\tau) < M\eta\right]\\
\leq\ &\Pr\left[\inf_{\|\varDelta\|=M,\,\tau\in[\underline{\tau},\overline{\tau}]} -\varDelta^\top R_+(\varDelta,\tau) < M\eta,\ \inf_{\|\varDelta\|=M,\,\tau\in[\underline{\tau},\overline{\tau}]} -\varDelta^\top\big(-D_+(\tau)\varDelta + R_+(0,\tau)\big) \geq 2M\eta\right]\\
&+\Pr\left[\inf_{\|\varDelta\|=M,\,\tau\in[\underline{\tau},\overline{\tau}]} -\varDelta^\top\big(-D_+(\tau)\varDelta + R_+(0,\tau)\big) < 2M\eta\right].
\end{aligned} \tag{S.53}$$

By the Cauchy–Schwarz inequality, on the first event on the right-hand side of (S.53), there exist $\varDelta_0$ with $\|\varDelta_0\|=M$ and $\tau_0\in[\underline{\tau},\overline{\tau}]$ such that

$$-M\|R_+(\varDelta_0,\tau_0)+D_+(\tau_0)\varDelta_0 - R_+(0,\tau_0)\| - \varDelta_0^\top\big(-D_+(\tau_0)\varDelta_0 + R_+(0,\tau_0)\big) < M\eta, \tag{S.54}$$

and hence

$$-M\|R_+(\varDelta_0,\tau_0)+D_+(\tau_0)\varDelta_0 - R_+(0,\tau_0)\| + \inf_{\|\varDelta\|=M,\,\tau\in[\underline{\tau},\overline{\tau}]}\left[-\varDelta^\top\big(-D_+(\tau)\varDelta + R_+(0,\tau)\big)\right] < M\eta. \tag{S.55}$$

Since the infimum on the left-hand side is no less than $2\eta M$ on this event, there exist $\varDelta_0$ with $\|\varDelta_0\|=M$ and $\tau_0\in[\underline{\tau},\overline{\tau}]$ such that

$$\eta \leq \|R_+(\varDelta_0,\tau_0)+D_+(\tau_0)\varDelta_0 - R_+(0,\tau_0)\|. \tag{S.56}$$

Therefore,

$$\begin{aligned}&\Pr\left[\inf_{\|\Delta\|=M,\,\tau\in[\underline{\tau},\overline{\tau}]} -\Delta^\top R_+(\Delta,\tau) < M\eta,\ \inf_{\|\Delta\|=M,\,\tau\in[\underline{\tau},\overline{\tau}]} -\Delta^\top\big(-D_+(\tau)\Delta + R_+(0,\tau)\big) \geq 2M\eta\right]\\ \leq\ &\Pr\left[\sup_{\|\Delta\|=M,\,\tau\in[\underline{\tau},\overline{\tau}]} \|R_+(\Delta,\tau)+D_+(\tau)\Delta - R_+(0,\tau)\| \geq \eta\right]. \end{aligned} \tag{S.57}$$

For all $\|\Delta\| = M$, $\tau\in[\underline{\tau},\overline{\tau}]$, we have

$$\Delta^\top D_+(\tau)\Delta \ \geq\ M^2\left\{\inf_{\tau\in[\underline{\tau},\overline{\tau}]} \operatorname{mineig}\left(D_+(\tau)\right)\right\}, \tag{S.58}$$

$$-\Delta^\top R_+(0,\tau) \ \geq\ -M\left\{\sup_{\tau\in[\underline{\tau},\overline{\tau}]} \|R_+(0,\tau)\|\right\}, \tag{S.59}$$

where

$$\inf_{\tau\in[\underline{\tau},\overline{\tau}]} \operatorname{mineig}\left(D_+(\tau)\right) = \varphi\left\{\inf_{\tau\in[\underline{\tau},\overline{\tau}]} \varphi_+(\tau)\right\}\operatorname{mineig}\left(\Lambda_{+,p}\right) > 0.$$

Hence

$$\begin{aligned}&\Pr\left[\inf_{\|\Delta\|=M,\,\tau\in[\underline{\tau},\overline{\tau}]} -\Delta^\top\big(-D_+(\tau)\Delta + R_+(0,\tau)\big) < 2M\eta\right]\\ \leq\ &\Pr\left[M^2\left\{\inf_{\tau\in[\underline{\tau},\overline{\tau}]} \operatorname{mineig}\left(D_+(\tau)\right)\right\} - M\left\{\sup_{\tau\in[\underline{\tau},\overline{\tau}]} \|R_+(0,\tau)\|\right\} < 2M\eta\right]. \end{aligned} \tag{S.60}$$

We have shown that for all $M>0$, $\eta>0$,

$$\begin{aligned}&\Pr\left[\inf_{\|\Delta\|=M,\,\tau\in[\underline{\tau},\overline{\tau}]} -\Delta^\top R_+(\Delta,\tau) < M\eta\right]\\ \leq\ &\Pr\left[\sup_{\|\Delta\|=M,\,\tau\in[\underline{\tau},\overline{\tau}]} \|R_+(\Delta,\tau)+D_+(\tau)\Delta - R_+(0,\tau)\| \geq \eta\right]\\ &+\Pr\left[\sup_{\tau\in[\underline{\tau},\overline{\tau}]} \|R_+(0,\tau)\| > M\left\{\inf_{\tau\in[\underline{\tau},\overline{\tau}]} \operatorname{mineig}\left(D_+(\tau)\right)\right\} - 2\eta\right]. \end{aligned} \tag{S.61}$$

By Lemma S.4, we have $\sup_{\tau\in[\underline{\tau},\overline{\tau}]} \|R_+(0,\tau)\| = O_p(1)$. It follows from this result, (S.61), and (S.32) that for all $\eta,\varepsilon>0$, there exists an $M>0$ such that, for all sufficiently large n,

$$\Pr\left[\inf_{\|\Delta\|=M,\,\tau\in[\underline{\tau},\overline{\tau}]} -\Delta^\top R_+(\Delta,\tau) < M\eta\right] < \varepsilon. \tag{S.62}$$

By (S.52), for all $\|\Delta\| = M$, $\tau\in[\underline{\tau},\overline{\tau}]$ and $\lambda\geq 1$,

$$-\left(\frac{\Delta}{M}\right)^\top R_+(\Delta,\tau) \leq -\left(\frac{\Delta}{M}\right)^\top R_+(\lambda\Delta,\tau) \leq \|R_+(\lambda\Delta,\tau)\|, \tag{S.63}$$

and therefore, for all $M > 0$ and $\eta > 0$,

$$\Pr\left[\inf_{\|\varDelta\|\geq M,\,\tau\in[\underline{\tau},\overline{\tau}]}\|R_{+}(\varDelta,\tau)\|<\eta\right]\leq\Pr\left[\inf_{\|\varDelta\|=M,\,\tau\in[\underline{\tau},\overline{\tau}]}-\varDelta^{\top}R_{+}(\varDelta,\tau)<M\eta\right]. \tag{S.64}$$

It follows from the above inequality and (S.62) that for all $\varepsilon,\eta>0$, there exists an $M>0$ such that, for all sufficiently large n,

$$\Pr\left[\inf_{\|\varDelta\|\geq M,\,\tau\in[\underline{\tau},\overline{\tau}]}\|R_{+}(\varDelta,\tau)\|<\eta\right]<\varepsilon. \tag{S.65}$$

Clearly, we have

$$\begin{aligned}
&\Pr\left[\sup_{\tau\in[\underline{\tau},\overline{\tau}]}\|\varDelta_{+}(\tau)\|>M\right]\\
\leq\ &\Pr\left[\sup_{\tau\in[\underline{\tau},\overline{\tau}]}\|\varDelta_{+}(\tau)\|>M,\ \sup_{\tau\in[\underline{\tau},\overline{\tau}]}\|R_{+}(\varDelta_{+}(\tau),\tau)\|<\eta\right]+\Pr\left[\sup_{\tau\in[\underline{\tau},\overline{\tau}]}\|R_{+}(\varDelta_{+}(\tau),\tau)\|\geq\eta\right]\\
\leq\ &\Pr\left[\inf_{\|\varDelta\|\geq M,\,\tau\in[\underline{\tau},\overline{\tau}]}\|R_{+}(\varDelta,\tau)\|<\eta\right]+\Pr\left[\sup_{\tau\in[\underline{\tau},\overline{\tau}]}\|R_{+}(\varDelta_{+}(\tau),\tau)\|\geq\eta\right].
\end{aligned} \tag{S.66}$$

It follows from the above inequality, (S.65) and the uniform-in-τ bound $\|R_{+}(\varDelta_{+}(\tau),\tau)\|=o_p(1)$ that $\varDelta_{+}(\tau)=O_p(1)$, uniformly in $\tau\in[\underline{\tau},\overline{\tau}]$.

For all $\varepsilon>0$ and $M>0$, by the union bound,

$$\begin{aligned}
&\Pr\left[\sup_{\tau\in[\underline{\tau},\overline{\tau}]}\|R_{+}(\varDelta_{+}(\tau),\tau)+D_{+}(\tau)\varDelta_{+}(\tau)-R_{+}(0,\tau)\|>\varepsilon\right]\\
\leq\ &\Pr\left[\sup_{\tau\in[\underline{\tau},\overline{\tau}]}\|\varDelta_{+}(\tau)\|>M\right]+\Pr\left[\sup_{\|\varDelta\|\leq M,\,\tau\in[\underline{\tau},\overline{\tau}]}\|R_{+}(\varDelta,\tau)+D_{+}(\tau)\varDelta-R_{+}(0,\tau)\|>\varepsilon\right].
\end{aligned} \tag{S.67}$$

Now it follows from this inequality, $\sup_{\tau\in[\underline{\tau},\overline{\tau}]}\|\varDelta_{+}(\tau)\|=O_p(1)$ and (S.32) that

$$R_{+}(\varDelta_{+}(\tau),\tau)+D_{+}(\tau)\varDelta_{+}(\tau)-R_{+}(0,\tau)=o_p(1),$$

uniformly in $\tau\in[\underline{\tau},\overline{\tau}]$. The conclusion follows from this result and the uniform-in-τ bound $\|R_{+}(\varDelta_{+}(\tau),\tau)\|=o_p(1)$. ■

Proof of Theorem 2. We have $\mathrm{e}_{p+1,1}^{\top}D_{+}^{-1}(\tau)\,K_{+,p}(X_i/h)=\bar{W}_{+,p,i}/\varphi_{+}(\tau)$ and $\mathrm{e}_{p+1,1}^{\top}\Lambda_{+,p}^{-1}\pi_{+}=\omega_{p,0}^{0,2}$. The second equality implies that $\mathrm{e}_{p+1,1}^{\top}D_{+}^{-1}(\tau)\,\pi_{+}\varphi/\omega_{p,0}^{0,2}=1/\varphi_{+}(\tau)$. Moreover, we have $\mathrm{e}_{p+1,1}^{\top}\Lambda_{+,p}^{-1}\int_0^1 K_{+,p}(u)\,u^{p+1}\mathrm{d}u=\omega_{+,p,0}^{p+1,1}$, so that $\mathrm{e}_{p+1,1}^{\top}D_{+}^{-1}(\tau)\,\bar{\mathscr{B}}_{+}(\tau)$ equals the quantity $\mathscr{B}_{+,p}(\tau)$

defined by (S.68) below. Then, by Lemmas S.3–S.5, we have

$$\begin{aligned}
\mathrm{e}_{p+1,1}^{\top} \mathit{\Delta}_{+}(\tau) &= \frac{1}{\sqrt{nh}} \sum_{i} \bar{W}_{+,p,i} \left(\frac{U_i(\tau)}{\varphi_{+}(\tau)} \right) - \left(\frac{\gamma_{+}(\tau)}{\varphi_{+}(\tau)} \right)^{\top} \left\{ \frac{1}{\sqrt{nh}} \sum_{i} \bar{W}_{p,i} \xi_i \right\} \\
&\quad + \sqrt{nh} h^{p+1} \mathscr{B}_{+,p}(\tau) + o_p(1), \text{ where} \\
\mathscr{B}_{+,p}(\tau) &:= \frac{\omega_{+,p,0}^{p+1,1} \kappa_{Y,+}^{(p+1)}(\tau)}{(p+1)!} - \left(\frac{\gamma_{+}(\tau)}{\varphi_{+}(\tau)} \right)^{\top} \left(\frac{\mu_{Z,+}^{(p+1)} \omega_{+,p,0}^{p+1,1} - \mu_{Z,-}^{(p+1)} \omega_{-,p,0}^{p+1,1}}{(p+1)!} \right), \quad \text{(S.68)}
\end{aligned}$$

uniformly in $\tau \in [\underline{\tau}, \overline{\tau}]$. By similar arguments, we obtain a linearization for $\mathrm{e}_{p+1,1}^{\top} \mathit{\Delta}_{-}(\tau)$. Combining these results and using (S.50), we get

$$\sqrt{nh} \left(\widehat{\vartheta}_{p}^{eb}(\tau) - \vartheta(\tau) - h^{p+1} \mathscr{B}_{\star,p}(\tau) \right) = \mathbb{Z}_{p,n}(\tau) + o_p(1), \quad \text{(S.69)}$$

uniformly in $\tau \in [\underline{\tau}, \overline{\tau}]$, where

$$\phi_i(\tau) := \left(\frac{\bar{W}_{+,p,i}}{\varphi_{+}(\tau)} - \frac{\bar{W}_{-,p,i}}{\varphi_{-}(\tau)} \right) U_i(\tau) - \bar{W}_{p,i} \xi_i^{\top} \gamma(\tau) \text{ and } \mathbb{Z}_{p,n}(\tau) := \frac{1}{\sqrt{nh}} \sum_{i} \phi_i(\tau). \quad \text{(S.70)}$$

We now upgrade the uniform linearization above to convergence in distribution in $\ell^{\infty}[\underline{\tau}, \overline{\tau}]$ via the changing-class Donsker theorem of van der Vaart (1998, Theorem 19.28). Define

$$\begin{aligned}
& g_7(y, x, z \mid \tau) := \\
& \quad \frac{1}{\varphi \sqrt{h}} \left\{ \left(\frac{\mathcal{K}_{+,p,0}(x/h) \mathbb{1}(x>0)}{\varphi_{+}(\tau)} - \frac{\mathcal{K}_{-,p,0}(x/h) \mathbb{1}(x<0)}{\varphi_{-}(\tau)} \right) \left\{ \tau - \mathbb{1}\left(y \leq Q_{Y|X}(\tau \mid x) \right) \right\} \right. \\
& \qquad \left. - \left(\mathcal{K}_{+,p,0}\left(\frac{x}{h}\right) \mathbb{1}(x>0) - \mathcal{K}_{-,p,0}\left(\frac{x}{h}\right) \mathbb{1}(x<0) \right) \left(z - \mu_Z(x) \right)^{\top} \gamma(\tau) \right\},
\end{aligned}$$

so that $g_7(Y_i, X_i, Z_i \mid \tau) = \phi_i(\tau)/\sqrt{h}$. Let $\mathscr{G}_7 := \{ g_7(\cdot \mid \tau) : \tau \in [\underline{\tau}, \overline{\tau}] \}$. Since $\mathrm{E}[U(\tau) \mid X] = 0$ and $\mathrm{E}[\xi \mid X] = 0$, $\mathrm{E}[\phi(\tau)] = 0$, and hence $\mathbb{G}_n g_7(\cdot \mid \tau) = \mathbb{Z}_{p,n}(\tau)$. We take the index-set semimetric to be the Euclidean metric, under which $[\underline{\tau}, \overline{\tau}]$ is totally bounded.

Define the envelope

$$\begin{aligned}
G_7(y, x, z) &:= \frac{1}{\varphi \sqrt{h}} \left(\left| \mathcal{K}_{+,p,0}\left(\frac{x}{h}\right) \right| + \left| \mathcal{K}_{-,p,0}\left(\frac{x}{h}\right) \right| \right) \\
&\quad \times \left\{ \left(\inf_{\tau \in [\underline{\tau}, \overline{\tau}]} \min\{ \varphi_{+}(\tau), \varphi_{-}(\tau) \} \right)^{-1} + \| z - \mu_Z(x) \| \sup_{\tau \in [\underline{\tau}, \overline{\tau}]} \| \gamma(\tau) \| \right\}. \quad \text{(S.71)}
\end{aligned}$$

Then $\sup_{\tau \in [\underline{\tau}, \overline{\tau}]} |g_7(\cdot \mid \tau)| \leq G_7$. A change of variables yields $\mathbb{P} G_7^2 = O(1)$ and $\mathbb{P} G_7^{2+\delta} = O\left(h^{-\delta/2}\right)$. By Markov's inequality,

$$\mathbb{P} G_7^2 \mathbb{1}\left(G_7 > \eta \sqrt{n} \right) \leq \mathbb{P} G_7^{2+\delta} \left(\eta \sqrt{n} \right)^{-\delta} = O\left((nh)^{-\delta/2} \right),$$

for every $\eta > 0$. This verifies the Lindeberg envelope condition.

The class (S.43) is VC-subgraph with index 2. By Kosorok (2007, Lemma 9.6), the function class $\left\{(x,z) \mapsto (z - \mu_Z(x))^\top b : b \in \mathbb{R}^{d_z}\right\}$ is VC-subgraph with index no more than $d_z + 2$. By Kosorok (2007, Theorem 9.3 and Lemma 9.9(vi)) and Chernozhukov et al. (2014, Lemma A.6), $\mathscr{G}_7$ is VC-type with the envelope G_7 and characteristics independent of n. It follows that $\mathscr{G}_7$ has a bounded uniform entropy integral. Hence the convergence condition for the uniform entropy integral in the statement of van der Vaart (1998, Theorem 19.28) is satisfied.

Next, we show that the covariance convergence condition is satisfied. For any $(\tau, \tau') \in [\underline{\tau}, \overline{\tau}]^2$, we write

$$
\begin{aligned}
&\mathbb{P}\left(g_7(\cdot \mid \tau)\, g_7(\cdot \mid \tau')\right) \\
=\;& \sum_{s \in \{+,-\}} \left\{ \mathrm{E}\left[\frac{1}{h}\left(\frac{\bar{W}_{s,p}^2}{\varphi_s(\tau)\,\varphi_s(\tau')}\right) U(\tau)\, U(\tau')\right] + \mathrm{E}\left[\frac{1}{h}\bar{W}_{s,p}^2 \left(\xi^\top \gamma(\tau)\right)\left(\xi^\top \gamma(\tau')\right)\right] \right. \\
&\left. - \mathrm{E}\left[\frac{1}{h}\left(\frac{\bar{W}_{s,p}^2}{\varphi_s(\tau)}\right) U(\tau)\, \xi^\top \gamma(\tau')\right] - \mathrm{E}\left[\frac{1}{h}\left(\frac{\bar{W}_{s,p}^2}{\varphi_s(\tau')}\right) U(\tau')\, \xi^\top \gamma(\tau)\right] \right\}.
\end{aligned}
$$

Note that $\mathrm{E}\left[U(\tau)\, U(\tau') \mid X\right] = \min\{\tau, \tau'\} - \tau\tau'$, $\mathrm{E}\left[\xi U(\tau) \mid X\right] = \mu_{ZU(\tau)}(X)$, and $\mathrm{E}\left[\xi\xi^\top \mid X\right] = \mathrm{Var}\left[Z \mid X\right]$. By a change of variables, $\mathbb{P}\left(g_7(\cdot \mid \tau)\, g_7(\cdot \mid \tau')\right)$ converges to $\mathscr{C}_{\star,p}(\tau, \tau')$.

Write $\phi(\tau) = \phi_u(\tau) - \phi_z(\tau)$ with $\phi_u(\tau) \coloneqq \left(\bar{W}_{+,p}/\varphi_+(\tau) - \bar{W}_{-,p}/\varphi_-(\tau)\right) U(\tau)$ and $\phi_z(\tau) \coloneqq \bar{W}_p \xi^\top \gamma(\tau)$. By adding and subtracting $\left(\bar{W}_{+,p}/\varphi_+(\tau) - \bar{W}_{-,p}/\varphi_-(\tau)\right) U(\tau')$ in $\phi_u(\tau) - \phi_u(\tau')$, we have

$$
\begin{aligned}
&\mathrm{E}\left[\frac{1}{h}\left(\phi_u(\tau) - \phi_u(\tau')\right)^2\right] \lesssim \\
&\sum_{s \in \{+,-\}} \left\{ \varphi_s^{-2}(\tau)\, \mathrm{E}\left[\frac{1}{h}\bar{W}_{s,p}^2 \left(U(\tau) - U(\tau')\right)^2\right] + \left(\varphi_s^{-1}(\tau) - \varphi_s^{-1}(\tau')\right)^2 \mathrm{E}\left[\frac{1}{h}\bar{W}_{s,p}^2 U^2(\tau')\right] \right\}. \quad \text{(S.72)}
\end{aligned}
$$

By (S.37), we have $\mathrm{E}\left[\left(U(\tau) - U(\tau')\right)^2 \mid X\right] \leq |\tau - \tau'|$. Combining this with $\inf_{\tau \in [\underline{\tau}, \overline{\tau}]} \varphi_s(\tau) > 0$, the continuity of φ_s^{-1}, $|U(\tau')| \leq 1$, $\mathrm{E}\left[h^{-1}\bar{W}_{s,p}^2\right] = O(1)$ and (S.72), we obtain

$$
\sup_{\tau, \tau' \in [\underline{\tau}, \overline{\tau}] : |\tau - \tau'| < \delta_n} \mathrm{E}\left[\frac{1}{h}\left(\phi_u(\tau) - \phi_u(\tau')\right)^2\right] = o(1), \quad \text{(S.73)}
$$

for every $\delta_n \downarrow 0$. Similarly, since

$$
\mathrm{E}\left[\frac{1}{h}\left(\phi_z(\tau) - \phi_z(\tau')\right)^2\right] \leq \left(\mathrm{E}\left[\frac{1}{h}\bar{W}_p^2 \|\xi\|^2\right]\right) \left\|\gamma(\tau) - \gamma(\tau')\right\|^2, \quad \text{(S.74)}
$$

the continuity of $\tau \mapsto \gamma(\tau)$ and $\mathrm{E}\left[h^{-1}\bar{W}_p^2 \|\xi\|^2\right] = O(1)$ yield that (S.73) also holds for ϕ_z. We have verified condition (19.27) of van der Vaart (1998) directly under the Euclidean metric.

Now the hypotheses of van der Vaart (1998, Theorem 19.28) are all verified for the changing class $\mathscr{G}_7$. Therefore, $\mathbb{Z}_{p,n} \rightsquigarrow \mathbb{Z}_p$ in $\ell^\infty[\underline{\tau}, \overline{\tau}]$, where $\mathbb{Z}_p$ is the centered tight Gaussian process with covariance kernel $\mathscr{C}_{\star,p}$. The first conclusion follows from this result, the uniform stochastic linearization (S.69) and Slutsky's lemma. The second conclusion follows from the first one and the continuous mapping theorem applied to the evaluation functional. ∎

S.3 Proof of Theorem 3

Let $\Pr_*[\cdot]$ denote probability under the resampling distribution (the conditional distribution given the original sample). The following notation follows Marmer and Shneyerov (2012). Let O_p^* and o_p^* denote the bootstrap analogues of O_p and o_p. For a sequence of random variables λ_n and a deterministic positive sequence α_n, we write $\lambda_n = O_p^*(\alpha_n)$ if for all $\varepsilon > 0$, there exists some $M > 0$ such that $\limsup_{n\uparrow\infty} \Pr[\Pr_*[|\lambda_n/\alpha_n| \geq M] > \varepsilon] < \varepsilon$. We write $\lambda_n = o_p^*(\alpha_n)$ if for all $\varepsilon > 0$, $\Pr_*[|\lambda_n/\alpha_n| > \varepsilon] \to_p 0$ as $n \uparrow \infty$ (equivalently, $\Pr_*[|\lambda_n/\alpha_n| > \varepsilon] < \varepsilon$ wpa1). The usual sum and product rules for O_p and o_p (e.g., van der Vaart, 1998, Section 2.2) hold for O_p^* and o_p^*. $\lambda_n = O_p^*(\alpha_n)$ for $\alpha_n \downarrow 0$ implies $\lambda_n = o_p^*(1)$. Further useful properties are summarized in the following lemma, which extends Marmer and Shneyerov (2012, Lemmas S.1 and S.2) slightly.

Lemma S.6. *(i) $\lambda_n = o_p(\alpha_n)$ implies $\lambda_n = o_p^*(\alpha_n)$. (ii) $\lambda_n = O_p(\alpha_n)$ implies $\lambda_n = O_p^*(\alpha_n)$. (iii) $\mathrm{E}_*[|\lambda_n|^q] = O_p(\alpha_n^q)$ for some $q \in [1,2]$ implies $\lambda_n = O_p^*(\alpha_n)$. (iv) $\lambda_n = o_p^*(1)$ implies $\mathrm{E}_*[|\lambda_n| \wedge M] = o_p(1)$, for any $M > 0$.*

Proof of Lemma S.6. By Markov's inequality, for all $\varepsilon, \delta > 0$,

$$\Pr[\Pr_*[|\lambda_n/\alpha_n| > \varepsilon] > \delta] \leq \frac{\Pr[|\lambda_n/\alpha_n| > \varepsilon]}{\delta}.$$

Part (i) follows from this result and $\Pr[|\lambda_n/\alpha_n| > \varepsilon] \to 0$ if $\lambda_n = o_p(\alpha_n)$. Similarly, by Markov's inequality, for all $M, \varepsilon > 0$,

$$\Pr[\Pr_*[|\lambda_n/\alpha_n| \geq M] > \varepsilon] \leq \frac{\Pr[|\lambda_n/\alpha_n| \geq M]}{\varepsilon}.$$

Part (ii) follows from this result by choosing $M > 0$ such that $\limsup_{n\uparrow\infty} \Pr[|\lambda_n/\alpha_n| \geq M] < \varepsilon^2$. Part (iii) follows from essentially the same arguments as those in the proof of Marmer and Shneyerov (2012, Lemma S.2). For Part (iv), note that for all $\varepsilon > 0$, $\mathrm{E}_*[|\lambda_n| \wedge M] \leq \varepsilon/2 + M \cdot \Pr_*[|\lambda_n| > \varepsilon/2]$. Now, $\lambda_n = o_p^*(1)$ implies that $\Pr_*[|\lambda_n| > \varepsilon/2] \leq \varepsilon/(2M)$ wpa1 and it follows that $\mathrm{E}_*[|\lambda_n| \wedge M] \leq \varepsilon$ wpa1. ∎

Let $\bar{W}^*_{+,p,i} := \mathrm{e}_{p+1,1}^\top (\varphi\Lambda_{+,p})^{-1} K_{+,p}(X_i^*/h)$ denote the bootstrap analogue of $\bar{W}_{+,p,i}$. Let $\Pi^*_{+,p}$ be defined by the right-hand side of (1) with X_i replaced by X_i^*. Similarly, let $\left(\bar{W}^*_{-,p,i}, \Pi^*_{-,p}\right)$ be the bootstrap analogues of $\left(\bar{W}_{-,p,i}, \Pi_{-,p}\right)$ and $\bar{W}^*_{p,i} := \bar{W}^*_{+,p,i} - \bar{W}^*_{-,p,i}$. By Lemma S.6(iii) and Markov's inequality, $\left\|\Pi^*_{+,p} - \Pi_{+,p}\right\| = o^*_p(1)$. By this result, $\|\Pi_{+,p} - \varphi\Lambda_{+,p}\| = o_p(1)$, and Lemma S.6(i), we have $\left\|\Pi^*_{+,p} - \varphi\Lambda_{+,p}\right\| = o^*_p(1)$. Now this result and (S.2) yield

$$\left\|\left(\Pi^*_{+,p}\right)^{-1} - \left(\varphi\Lambda_{+,p}\right)^{-1}\right\| = o^*_p(1). \tag{S.75}$$

Similar results hold for $\Pi^*_{-,p}$. The following result is a bootstrap analogue of Lemma S.1.

Lemma S.7. *Let $\{(A^*_i, X^*_i)\}_{i=1}^n$ denote a bootstrap sample from $\{(A_i, X_i)\}_{i=1}^n$. Suppose that all assumptions and conditions in the statement of Lemma S.1 hold. Then, for every $k \in \mathbb{N}$, the following results hold, with δ and r as defined there. (i)*

$$\mathrm{E}_*\left[\frac{1}{h}\left(\bar{W}^*_{+,p}\right)^k A^*\right] = \frac{\mu_{A,+}\omega^{0,k}_{+,p,0}}{\varphi^{k-1}} + o_p(1).$$

(ii)

$$\frac{1}{nh}\sum_i W^*_{+,p,i}\mu_A\left(X^*_i\right) = \mu_{A,+} + \frac{\mu^{(p+1)}_{A,+}}{(p+1)!}\omega^{p+1,1}_{+,p,0}h^{p+1} + o^*_p\left(h^{p+1}\right).$$

(iii)

$$\frac{1}{nh}\sum_i \left(W^*_{+,p,i}\right)^k A^*_i - \mathrm{E}_*\left[\frac{1}{h}\left(\bar{W}^*_{+,p}\right)^k A^*\right] = o^*_p(1).$$

(iv) $\max_i \left|W^*_{+,p,i}A^*_i\right| = o^*_p\left((nh)^{1/r}\right)$. *Similar results with "+" replaced by "−" also hold.*

Proof of Lemma S.7. Note that the resampling distribution of (A^*, X^*) is the empirical distribution of $\{(A_i, X_i)\}_{i=1}^n$. It follows from (S.10) and Lemma S.1(iii) that

$$\mathrm{E}_*\left[\frac{1}{h}\left(\bar{W}^*_{+,p}\right)^k A^*\right] - \mathrm{E}\left[\frac{1}{h}\bar{W}^k_{+,p}A\right] = \frac{1}{nh}\sum_i \bar{W}^k_{+,p,i}A_i - \mathrm{E}\left[\frac{1}{h}\bar{W}^k_{+,p}A\right] = o_p(1).$$

Part (i) follows from this result and Lemma S.1(i).

By Lemma S.6(iii) and Markov's inequality, $(nh)^{-1}\sum_i \left\|K_{+,p}\left(X^*_i/h\right)\left(X^*_i/h\right)^{p+1}\right\| = O^*_p(1)$. It then follows from this result and (S.75) that

$$\frac{1}{nh}\sum_i \left|W^*_{+,p,i}\left(\frac{X^*_i}{h}\right)^{p+1}\right| \le \left\|\left(\Pi^*_{+,p}\right)^{-1}\right\|\left(\frac{1}{nh}\sum_i \left\|K_{+,p}\left(\frac{X^*_i}{h}\right)\left(\frac{X^*_i}{h}\right)^{p+1}\right\|\right) = O^*_p(1).$$

By (S.75), Lemma S.6(i), and Lemma S.6(iii),

$$\begin{aligned}\frac{1}{nh}\sum_i W^*_{+,p,i}\left(\frac{X^*_i}{h}\right)^{p+1} &= \mathrm{e}^\top_{p+1,1}\left\{\left(\varphi\Lambda_{+,p}\right)^{-1} + o^*_p(1)\right\}\left(\varphi\int K_{+,p}(u)\,u^{p+1}\mathrm{d}u + o^*_p(1)\right) \\ &= \omega^{p+1,1}_{+,p,0} + o^*_p(1).\end{aligned}$$

Then, the conclusion in Part (ii) follows from these results and a Taylor expansion similar to that in (S.5).

To prove Part (iii), we use a decomposition similar to (S.9):

$$\frac{1}{nh}\sum_i \left(W^*_{+,p,i}\right)^k A^*_i - \mathrm{E}_*\left[\frac{1}{h}\left(\bar{W}^*_{+,p}\right)^k A^*\right] = \frac{1}{nh}\sum_i \left(\left(W^*_{+,p,i}\right)^k - \left(\bar{W}^*_{+,p,i}\right)^k\right) A^*_i + \left\{\frac{1}{nh}\sum_i \left(\bar{W}^*_{+,p,i}\right)^k A^*_i - \mathrm{E}_*\left[\frac{1}{h}\left(\bar{W}^*_{+,p}\right)^k A^*\right]\right\}. \tag{S.76}$$

From arguments similar to those used for the first term on the right-hand side of (S.9), together with (S.75), Lemma S.6(iii), and Markov's inequality, the first term on the right-hand side is $o^*_p(1)$. Then, we can obtain a bootstrap analogue of the inequality in (S.10) by using the same arguments. By computing the population moment and applying Markov's inequality, $(nh)^{-1}\sum_i \left|\bar{W}^k_{+,p,i}A_i\right|^{1+\delta} = O_p(1)$. Now we have

$$\mathrm{E}_*\left[\left|\frac{1}{nh}\sum_i \left(\bar{W}^*_{+,p,i}\right)^k A^*_i - \mathrm{E}_*\left[\frac{1}{h}\left(\bar{W}^*_{+,p}\right)^k A^*\right]\right|^{1+\delta}\right] = O_p\left((nh)^{-\delta}\right),$$

and by Lemma S.6(iii), the second term on the right-hand side of (S.76) is also $o^*_p(1)$.

For Part (iv), note that since every bootstrap observation is one of the original observations, $\max_i \|K_{+,p}(X^*_i/h)\| |A^*_i|$ is bounded from above by $\max_i \|K_{+,p}(X_i/h)\| |A_i| = o_p\left((nh)^{1/r}\right)$. The conclusion follows from this observation, Lemma S.6(i), and (S.75). ■

Let $\widehat{\kappa}^*_{+,p}(\tau)$ and $R^*_+(\Delta,\tau)$ denote the bootstrap analogues of $\widehat{\kappa}_{+,p}(\tau)$ and $R_+(\Delta,\tau)$. The balancing weights are taken to be $\widetilde{w}^*_{p,i}$. Let $\tilde{Y}^*_{+,i}$ be the bootstrap analogue of $\tilde{Y}_{+,i}$. Let $\mathbb{P}^*_n g \coloneqq n^{-1}\sum_i g(Y^*_i, X^*_i, Z^*_i)$ denote the average under the empirical distribution of the bootstrap sample. Let $\mathbb{G}^*_n g \coloneqq \sqrt{n}\left(\mathbb{P}^*_n - \mathbb{P}_n\right) g$. The following result is the bootstrap analogue of Lemma S.3.

Lemma S.8. *Suppose that the assumptions in the statement of Theorem 3 hold. For all $M > 0$,*

$$\sup_{\|\Delta\|\le M,\, \tau\in[\underline{\tau},\overline{\tau}]} \left\|R^*_+(\Delta,\tau) + D_+(\tau)\Delta - R^*_+(0,\tau)\right\| = o^*_p(1). \tag{S.77}$$

Proof. We have

$$\widetilde{w}^*_{p,i} = \frac{1}{n}\cdot\frac{1+V^{*\top}_{p,i}\widetilde{\lambda}^*_p}{1+\left(n^{-1}\sum_j V^*_{p,j}\right)^\top \widetilde{\lambda}^*_p}, \text{ where}$$

$$\widetilde{\lambda}^*_p \coloneqq -\left(\sum_i V^*_{p,i}V^{*\top}_{p,i}\right)^{-1}\left(\sum_i V^*_{p,i}\right).$$

By Lemma S.7, we have

$$\begin{aligned}\frac{1}{nh}\sum_i V_{p,i}^* V_{p,i}^{*\top} &= \left(\frac{\omega_{p,0}^{0,2}}{\varphi}\right)\mu_{\bar{Z}\bar{Z}^\top,\pm} + o_p^*(1)\\ \frac{1}{nh}\sum_i W_{p,i}^* Z_i^* &= \frac{1}{nh}\sum_i W_{p,i}^*\xi_i^* + \frac{\left(\mu_{Z,+}^{(p+1)}\omega_{+,p,0}^{p+1,1} - \mu_{Z,-}^{(p+1)}\omega_{-,p,0}^{p+1,1}\right)}{(p+1)!}h^{p+1} + o_p^*\left(h^{p+1}\right),\end{aligned} \tag{S.78}$$

where $\xi_i^* \coloneqq Z_i^* - \mu_Z(X_i^*)$. By a simple variance calculation with respect to the resampling distribution, Lemma S.6(iii), and Markov's inequality,

$$\frac{1}{\sqrt{nh}}\sum_i K_{s,p}\left(\frac{X_i^*}{h}\right)\xi_i^{*\top} - \frac{1}{\sqrt{nh}}\sum_i K_{s,p}\left(\frac{X_i}{h}\right)\xi_i^\top = O_p^*(1),$$

for $s \in \{-,+\}$. Now, $(nh)^{-1/2}\sum_i K_{s,p}(X_i^*/h)\xi_i^{*\top} = O_p^*(1)$ follows from this result, the fact that $(nh)^{-1/2}\sum_i K_{s,p}(X_i/h)\xi_i^\top = O_p(1)$ and Lemma S.6(ii). Then, it follows from this result, (S.75), and the bootstrap analogue of the equality in (S.15) that $(nh)^{-1/2}\sum_i W_{p,i}^*\xi_i^* = O_p^*(1)$. Since $\sum_i W_{p,i}^* = 0$, the bootstrap analogue of (S.16) holds. By this result and (S.78), we have $(nh)^{-1/2}\sum_i V_{p,i}^* = O_p^*(1)$, $\widetilde{\lambda}_p^* = O_p^*\left((nh)^{-1/2}\right)$, and $\widetilde{w}_{p,i}^* = n^{-1}\left(1 + V_{p,i}^{*\top}\widetilde{\lambda}_p^*\right)\left(1 + O_p^*\left(n^{-1}\right)\right)$.

Decompose

$$R_+^*(\varDelta,\tau) - R_+^*(0,\tau) = T_1^*(\varDelta,\tau)\left\{1 + O_p^*\left(n^{-1}\right)\right\} + T_2^*(\varDelta,\tau)\left\{1 + O_p^*\left(n^{-1}\right)\right\},$$

where

$$\begin{aligned}T_1^*(\varDelta,\tau) &\coloneqq \frac{1}{\sqrt{nh}}\sum_i K_{+,p}\left(\frac{X_i^*}{h}\right)\left\{\mathbb{1}\left(\tilde{Y}_{+,i}^* \le 0\right) - \mathbb{1}\left(\tilde{Y}_{+,i}^* \le \frac{r_p^\top(X_i^*/h)\varDelta}{\sqrt{nh}}\right)\right\}\\ T_2^*(\varDelta,\tau) &\coloneqq \frac{1}{\sqrt{nh}}\sum_i K_{+,p}\left(\frac{X_i^*}{h}\right)\left(V_{p,i}^{*\top}\widetilde{\lambda}_p^*\right)\left\{\mathbb{1}\left(\tilde{Y}_{+,i}^* \le 0\right) - \mathbb{1}\left(\tilde{Y}_{+,i}^* \le \frac{r_p^\top(X_i^*/h)\varDelta}{\sqrt{nh}}\right)\right\}\end{aligned}$$

denote the bootstrap analogues of $T_1(\varDelta,\tau)$ and $T_2(\varDelta,\tau)$. Decomposing $T_1^*(\varDelta,\tau)$ as in (S.33), we have $\mathrm{E}_*[T_1^*(\varDelta,\tau)] = T_1(\varDelta,\tau)$. We can write the j-th coordinate of $T_1^*(\varDelta,\tau) - \mathrm{E}_*[T_1^*(\varDelta,\tau)]$ as $\mathbb{G}_n^* g_1(\cdot \mid \varDelta,\tau)$. Applying Chen and Kato (2020, Corollary 5.5) to $\mathrm{E}_*\left[\|\mathbb{G}_n^*\|_{\mathscr{G}_1}\right]$, we have

$$\mathrm{E}_*\left[\|\mathbb{G}_n^*\|_{\mathscr{G}_1}\right] \lesssim \sqrt{\left(\sup_{\|\varDelta\|\le M,\,\tau\in[\underline{\tau},\overline{\tau}]}\mathbb{P}_n g_1^2(\cdot \mid \varDelta,\tau)\right)\log(n)} + \frac{\log(n)}{\sqrt{nh}}. \tag{S.79}$$

Let $\mathscr{G}_1^{(2)} \coloneqq \left\{g_1^2(\cdot \mid \varDelta,\tau) : \|\varDelta\| \le M,\, \tau \in [\underline{\tau},\overline{\tau}]\right\}$. By Chernozhukov et al. (2014, Corollary A.1), $\mathscr{G}_1^{(2)}$ is VC-type with a constant envelope G_1^2 and characteristics independent of n. By (S.37) and calculations as in (S.38), we have $\mathbb{P}g_1^4(\cdot \mid \varDelta,\tau) = O\left(\left(nh^3\right)^{-1/2}\right)$, uniformly in $\|\varDelta\| \le M, \tau \in [\underline{\tau},\overline{\tau}]$. Then, it follows from this result and Chen and Kato (2020, Corollary 5.5) that $\|\mathbb{P}_n - \mathbb{P}\|_{\mathscr{G}_1^{(2)}} =$

$O_p\left(\sqrt{\log(n)}/(nh)^{3/4}\right)$. It now follows that $\mathbb{P}_n g_1^2(\cdot \mid \Delta,\tau) = O_p\left((nh)^{-1/2}\right)$, uniformly in $\|\Delta\| \leq M, \tau \in [\underline{\tau},\overline{\tau}]$. By this result and (S.79), we have $\mathrm{E}_*\left[\|\mathbb{G}_n^*\|_{\mathscr{G}_1}\right] = O_p\left(\sqrt{\log(n)}/(nh)^{1/4}\right)$ and $\|\mathbb{G}_n^*\|_{\mathscr{G}_1} = o_p^*(1)$ by Lemma S.6(iii). This gives that $T_1^*(\Delta,\tau) - \mathrm{E}_*[T_1^*(\Delta,\tau)] = o_p^*(1)$, uniformly in $\|\Delta\| \leq M, \tau \in [\underline{\tau},\overline{\tau}]$. By this result, (S.40), and Lemma S.6(i), we have $T_1^*(\Delta,\tau) = -D_+(\tau)\Delta + o_p^*(1)$, uniformly for all $\|\Delta\| \leq M, \tau \in [\underline{\tau},\overline{\tau}]$.

For $T_2^*(\Delta,\tau)$, we have

$$\sup_{\|\Delta\|\leq M} \|T_2^*(\Delta,\tau)\| \leq \left(\max_i \left|V_{p,i}^{*\top}\widetilde{\lambda}_p^*\right|\right)\left(\sqrt{n}\cdot \mathbb{P}_n^* g_2(\cdot \mid \tau)\right).$$

By Lemma S.7(iv), and $\widetilde{\lambda}_p^* = O_p^*\left((nh)^{-1/2}\right)$, we have $\max_i \left|V_{p,i}^{*\top}\widetilde{\lambda}_p^*\right| = o_p^*(1)$. Decompose $\sqrt{n}\cdot \mathbb{P}_n^* g_2(\cdot \mid \tau) = \sqrt{n}\cdot \mathbb{P}_n g_2(\cdot \mid \tau) + \mathbb{G}_n^* g_2(\cdot \mid \tau)$. We showed in the proof of Lemma S.3 that $\sqrt{n}\cdot \mathbb{P}_n g_2(\cdot \mid \tau) = O_p(1)$ uniformly in $\tau \in [\underline{\tau},\overline{\tau}]$. Let $\mathscr{G}_2^{(2)} \coloneqq \left\{g_2^2(\cdot \mid \tau) : \tau \in [\underline{\tau},\overline{\tau}]\right\}$. By Chernozhukov et al. (2014, Corollary A.1), $\mathscr{G}_2^{(2)}$ is VC-type with a constant envelope $G_2^2 \lesssim h^{-1}$ and characteristics independent of n. By (S.37), the LIE, a bound analogous to (S.34) and a change of variables, as in (S.38), we have $\mathbb{P}g_2^4(\cdot \mid \tau) = O\left((nh^3)^{-1/2}\right)$, uniformly in $\tau \in [\underline{\tau},\overline{\tau}]$. By Chen and Kato (2020, Corollary 5.5), $\|\mathbb{P}_n - \mathbb{P}\|_{\mathscr{G}_2^{(2)}} = o_p\left((nh)^{-1/2}\right)$, and since $\sup_{\tau\in[\underline{\tau},\overline{\tau}]} \mathbb{P}g_2^2(\cdot \mid \tau) = O\left((nh)^{-1/2}\right)$ as shown in the proof of Lemma S.3, we have $\sup_{\tau\in[\underline{\tau},\overline{\tau}]} \mathbb{P}_n g_2^2(\cdot \mid \tau) = O_p\left((nh)^{-1/2}\right)$. Applying Chen and Kato (2020, Corollary 5.5) under the resampling distribution, as in (S.79), we have $\mathrm{E}_*\left[\|\mathbb{G}_n^*\|_{\mathscr{G}_2}\right] = o_p(1)$, and $\|\mathbb{G}_n^*\|_{\mathscr{G}_2} = o_p^*(1)$ by Lemma S.6(iii). It follows from the above results and Lemma S.6(ii) that $\sqrt{n}\cdot \mathbb{P}_n^* g_2(\cdot \mid \tau) = O_p^*(1)$, uniformly in $\tau \in [\underline{\tau},\overline{\tau}]$. Now we have $T_2^*(\Delta,\tau) = o_p^*(1)$, uniformly in $\|\Delta\| \leq M, \tau \in [\underline{\tau},\overline{\tau}]$. ■

The following result is the bootstrap analogue of Lemma S.4.

Lemma S.9. *Suppose that the assumptions in the statement of Theorem 3 hold. We have the following linearization for $R_+^*(0,\tau)$:*

$$\begin{aligned} R_+^*(0,\tau) &= \frac{1}{\sqrt{nh}}\sum_i K_{+,p}\left(\frac{X_i^*}{h}\right)U_i^*(\tau) - \left(\frac{\pi_+\varphi}{\omega_{p,0}^{0,2}}\right)\gamma_+^\top(\tau)\left\{\frac{1}{\sqrt{nh}}\sum_i W_{p,i}^*\xi_i^*\right\} \\ &\quad + \bar{\mathscr{B}}_+(\tau)\sqrt{nh}h^{p+1} + o_p^*(1), \end{aligned}$$

uniformly in $\tau \in [\underline{\tau},\overline{\tau}]$, where $U_i^*(\tau) \coloneqq \tau - \mathbb{1}\left(Y_i^* \leq Q_{Y|X}(\tau \mid X_i^*)\right)$.

Proof of Lemma S.9. Decompose

$$\begin{aligned} R_+^*(0,\tau) &= \sqrt{\frac{n}{h}}\sum_i \widetilde{w}_{p,i}^* K_{+,p}\left(\frac{X_i^*}{h}\right)\psi_\tau\left(\tilde{Y}_{+,i}^*\right) \\ &= T_1^*(\tau)\left(1 + O_p^*\left(n^{-1}\right)\right) + T_2^*(\tau)\left(1 + O_p^*\left(n^{-1}\right)\right) \end{aligned} \tag{S.80}$$

as in the proof of Lemma S.4, where

$$
\begin{aligned}
T_1^*(\tau) &\coloneqq \frac{1}{\sqrt{nh}} \sum_i K_{+,p}\left(\frac{X_i^*}{h}\right) \psi_\tau\left(\tilde{Y}_{+,i}^*\right) \\
&= \frac{1}{\sqrt{nh}} \sum_i K_{+,p}\left(\frac{X_i^*}{h}\right) U_i^*(\tau) + T_3^*(\tau) \\
T_2^*(\tau) &\coloneqq \frac{1}{\sqrt{nh}} \sum_i K_{+,p}\left(\frac{X_i^*}{h}\right) \psi_\tau\left(\tilde{Y}_{+,i}^*\right) V_{p,i}^{*\top} \widetilde{\lambda}_p^* \\
T_3^*(\tau) &\coloneqq \frac{1}{\sqrt{nh}} \sum_i K_{+,p}\left(\frac{X_i^*}{h}\right) \left\{\mathbb{1}\left(Y_i^* \leq Q_{Y|X}(\tau \mid X_i^*)\right) - \mathbb{1}\left(Y_i^* \leq r_p^\top(X_i^*)\, \kappa_+(\tau)\right)\right\}.
\end{aligned}
$$

Since $\mathbb{P} g_3(\cdot \mid \tau) = 0$, we can write

$$
\frac{1}{\sqrt{nh}} \sum_i K_{+,p}\left(\frac{X_i^*}{h}\right) U_i^*(\tau) = \mathbb{G}_n^* g_3(\cdot \mid \tau) + \mathbb{G}_n g_3(\cdot \mid \tau).
$$

By van der Vaart and Wellner (1996, Theorem 2.14.1), $\mathrm{E}_*\left[\|\mathbb{G}_n^*\|_{\mathscr{G}_3}\right] \lesssim \sqrt{\mathbb{P}_n G_3^2} = O_p(1)$. It follows from this result, Lemma S.6(ii, iii), and (S.44) that

$$
\frac{1}{\sqrt{nh}} \sum_i K_{+,p}\left(\frac{X_i^*}{h}\right) U_i^*(\tau) = O_p^*(1), \tag{S.81}
$$

uniformly in $\tau \in [\underline{\tau}, \overline{\tau}]$.

We write $T_3^*(\tau) = (T_3^*(\tau) - \mathrm{E}_*[T_3^*(\tau)]) + \mathrm{E}_*[T_3^*(\tau)]$, where $\mathrm{E}_*[T_3^*(\tau)] = T_3(\tau)$ and $T_3(\tau)$ is defined by (S.42). We can also write the j-th coordinate of $T_3^*(\tau) - \mathrm{E}_*[T_3^*(\tau)]$ as $\mathbb{G}_n^* g_4(\cdot \mid \tau)$. Let $\mathscr{G}_8 \coloneqq \{g_4(\cdot \mid \tau) : \tau \in [\underline{\tau}, \overline{\tau}]\}$. The class $\mathscr{G}_8$ is VC-type with characteristics independent of n and a constant envelope proportional to $h^{-1/2}$. By Chen and Kato (2020, Corollary 5.5), we have

$$
\mathrm{E}_*\left[\|\mathbb{G}_n^*\|_{\mathscr{G}_8}\right] \lesssim \sqrt{\left(\sup_{\tau \in [\underline{\tau}, \overline{\tau}]} \mathbb{P}_n g_4^2(\cdot \mid \tau)\right) \log(n)} + \frac{\log(n)}{\sqrt{nh}}. \tag{S.82}
$$

Let $\mathscr{G}_4^{(2)} \coloneqq \left\{g_4^2(\cdot \mid \tau) : \tau \in [\underline{\tau}, \overline{\tau}]\right\}$. The class $\mathscr{G}_4^{(2)}$ is VC-type with characteristics independent of n and a constant envelope proportional to h^{-1}. By Chen and Kato (2020, Corollary 5.5),

$$
\mathrm{E}\left[\|\mathbb{G}_n\|_{\mathscr{G}_4^{(2)}}\right] \lesssim \sqrt{\left(\sup_{\tau \in [\underline{\tau}, \overline{\tau}]} \mathbb{P} g_4^4(\cdot \mid \tau)\right) \log(n)} + \frac{\log(n)}{\sqrt{nh^2}}.
$$

By calculations using the LIE, (S.45) and (S.37), we have $\mathbb{P} g_4^4(\cdot \mid \tau) = O(h^p)$, uniformly in $\tau \in [\underline{\tau}, \overline{\tau}]$. It follows that

$$
\|\mathbb{P}_n - \mathbb{P}\|_{\mathscr{G}_4^{(2)}} = O_p\left(\sqrt{\frac{h^p \log(n)}{n}} + \frac{\log(n)}{nh}\right).
$$

By this result, (S.46), and (S.82), we have $\mathrm{E}_*\left[\|\mathbb{G}_n^*\|_{\mathscr{G}_8}\right] = o_p(1)$ and $\|\mathbb{G}_n^*\|_{\mathscr{G}_8} = o_p^*(1)$. This shows that $T_3^*(\tau) - \mathrm{E}_*[T_3^*(\tau)] = o_p^*(1)$, uniformly in $\tau \in [\underline{\tau}, \overline{\tau}]$. In the proof of Lemma S.4, we showed that

$$T_3(\tau) = \sqrt{nh}h^{p+1}\left\{\left(\frac{\int_0^1 K_{+,p}(u)\, u^{p+1}\mathrm{d}u}{(p+1)!}\right)\varphi_+(\tau)\,\kappa_{Y,+}^{(p+1)}(\tau)\,\varphi + o(1)\right\} + o_p(1),$$

uniformly in $\tau \in [\underline{\tau}, \overline{\tau}]$. It now follows from these results and Lemma S.6(i) that

$$T_3^*(\tau) = \sqrt{nh}h^{p+1}\left\{\left(\frac{\int_0^1 K_{+,p}(u)\, u^{p+1}\mathrm{d}u}{(p+1)!}\right)\varphi_+(\tau)\,\kappa_{Y,+}^{(p+1)}(\tau)\,\varphi + o(1)\right\} + o_p^*(1), \tag{S.83}$$

uniformly in $\tau \in [\underline{\tau}, \overline{\tau}]$.

By (S.75) and Lemma S.6(iii), we have a bootstrap analogue of (S.48). Write

$$\frac{1}{nh}\sum_i \tilde{K}_{+,p}\left(\frac{X_i^*}{h}\right)\psi_\tau\left(\tilde{Y}_{+,i}^*\right)\bar{Z}_i^{*\top} = T_6^*(\tau) + T_7^*(\tau),$$

where $T_6^*(\tau)$ and $T_7^*(\tau)$ are bootstrap analogues of $T_6(\tau)$ and $T_7(\tau)$ defined by (S.49). Write $T_6^*(\tau) = (T_6^*(\tau) - \mathrm{E}_*[T_6^*(\tau)]) + \mathrm{E}_*[T_6^*(\tau)]$, where $\mathrm{E}_*[T_6^*(\tau)] = T_6(\tau)$. The jk-th coordinate of $T_6^*(\tau) - \mathrm{E}_*[T_6^*(\tau)]$ can be written as $\mathbb{G}_n^* g_6(\cdot \mid \tau)/\sqrt{nh}$. By van der Vaart and Wellner (1996, Theorem 2.14.1) and Markov's inequality, $\mathrm{E}_*\left[\|\mathbb{G}_n^*\|_{\mathscr{G}_6}\right] = O_p(1)$. It follows that $T_6^*(\tau) - \mathrm{E}_*[T_6^*(\tau)] = o_p^*(1)$, uniformly in $\tau \in [\underline{\tau}, \overline{\tau}]$. By similar arguments, $T_7^*(\tau) - \mathrm{E}_*[T_7^*(\tau)] = o_p^*(1)$, uniformly in $\tau \in [\underline{\tau}, \overline{\tau}]$. In the proof of Lemma S.4, we showed that $T_6(\tau) = \pi_+\varphi\mu_{U(\tau)\bar{Z}^\top,+} + o_p(1)$ and $\mathrm{E}_*[T_7^*(\tau)] = T_7(\tau) = o_p(1)$, uniformly in $\tau \in [\underline{\tau}, \overline{\tau}]$. Now, by Lemma S.6(i), $T_6^*(\tau) = \pi_+\varphi\mu_{U(\tau)\bar{Z}^\top,+} + o_p^*(1)$, uniformly in $\tau \in [\underline{\tau}, \overline{\tau}]$. Now we have

$$\frac{1}{nh}\sum_i K_{+,p}\left(\frac{X_i^*}{h}\right)\psi_\tau\left(\tilde{Y}_{+,i}^*\right)V_{p,i}^{*\top} = \pi_+\mu_{U(\tau)\bar{Z}^\top,+} + o_p^*(1),$$

uniformly in $\tau \in [\underline{\tau}, \overline{\tau}]$. By this result and (S.78), we have

$$T_2^*(\tau) = -\left(\frac{\pi_+\varphi}{\omega_{p,0}^{0,2}}\right)\gamma_+^\top(\tau)\left\{\frac{1}{\sqrt{nh}}\sum_i W_{p,i}^*\xi_i^* + \frac{\left(\mu_{Z,+}^{(p+1)}\omega_{+,p,0}^{p+1,1} - \mu_{Z,-}^{(p+1)}\omega_{-,p,0}^{p+1,1}\right)}{(p+1)!}\sqrt{nh}h^{p+1}\right\} + o_p^*(1),$$

uniformly in $\tau \in [\underline{\tau}, \overline{\tau}]$. The conclusion follows from this result, (S.80), and (S.83). ∎

Let $\mathit{\Delta}_+^*(\tau) \coloneqq \sqrt{nh}\mathrm{H}\left(\widehat{\kappa}_{+,p}^*(\tau) - \kappa_+(\tau)\right)$ and let $\mathit{\Delta}_-^*(\tau)$ be defined analogously. Then, we have

$$\sqrt{nh}\left(\widehat{\vartheta}_p^*(\tau) - \widehat{\vartheta}_p^{eb}(\tau)\right) = \mathrm{e}_{p+1,1}^\top\left(\mathit{\Delta}_+^*(\tau) - \mathit{\Delta}_-^*(\tau)\right) - \mathrm{e}_{p+1,1}^\top\left(\mathit{\Delta}_+(\tau) - \mathit{\Delta}_-(\tau)\right). \tag{S.84}$$

The following result is a bootstrap analogue of Lemma S.5. A similar result holds for $\mathit{\Delta}_-^*(\tau)$.

Lemma S.10. *Suppose that the assumptions in the statement of Theorem 3 hold. We have the* *following linearization:*

$$\varDelta_+^*(\tau) = D_+^{-1}(\tau) R_+^*(0,\tau) + o_p^*(1),$$

uniformly in $\tau \in [\underline{\tau}, \overline{\tau}]$.

Proof of Lemma S.10. Since we obtained $\max_i \left| V_{p,i}^{*\top} \widetilde{\lambda}_p^* \right| = o_p^*(1)$ and

$$\widetilde{w}_{p,i}^* = n^{-1} \left(1 + V_{p,i}^{*\top} \widetilde{\lambda}_p^*\right) \left(1 + O_p^*\left(n^{-1}\right)\right)$$

in the proof of Lemma S.8, we have $\max_i \left| \widetilde{w}_{p,i}^* \right| = O_p^*\left(n^{-1}\right)$. Moreover,

$$\Pr_*\left[\widetilde{w}_{p,i}^* > 0, \, \forall i\right] \geq \Pr_*\left[\max_i \left| V_{p,i}^{*\top} \widetilde{\lambda}_p^* \right| < \frac{1}{2}\right] \to_p 1.$$

When $\widetilde{w}_{p,i}^* > 0$ for all i, by a slight adaptation of Cai and Xu (2008, Lemma A.2) to the bootstrap sample,

$$\left\| \sum_i \widetilde{w}_{p,i}^* K_{+,p}\left(\frac{X_i^*}{h}\right) \psi_\tau\left(Y_i^* - r_p^\top\left(X_i^*\right) \widehat{\kappa}_{+,p}^*(\tau)\right) \right\| \leq (p+1)\, E_n \max_i \left\| \widetilde{w}_{p,i}^* K_{+,p}\left(\frac{X_i^*}{h}\right) \right\|,$$

where E_n is the maximal multiplicity of observations in the bootstrap sample. By Motwani and Raghavan (1995, Theorem 3.1), $E_n = o_p(\log(n))$. By this result and the union bound, for all $\varepsilon, M > 0$,

$$\begin{aligned} &\Pr\left[\Pr_*\left[\frac{n}{\log(n)} \left\| \sum_i \widetilde{w}_{p,i}^* K_{+,p}\left(\frac{X_i^*}{h}\right) \psi_\tau\left(Y_i^* - r_p^\top\left(X_i^*\right) \widehat{\kappa}_{+,p}^*(\tau)\right) \right\| \geq M\right] > \varepsilon\right] \\ &\quad \leq \Pr\left[\Pr_*\left[\frac{nE_n(p+1)}{\log(n)} \max_i \left\| \widetilde{w}_{p,i}^* K_{+,p}\left(\frac{X_i^*}{h}\right) \right\| > M\right] > \frac{\varepsilon}{2}\right] + \Pr\left[\Pr_*\left[\exists i : \widetilde{w}_{p,i}^* \leq 0\right] > \frac{\varepsilon}{2}\right]. \end{aligned}$$

By $\max_i \left| \widetilde{w}_{p,i}^* \right| = O_p^*\left(n^{-1}\right)$, $E_n = o_p(\log(n))$ and Lemma S.6(i), one can choose $M > 0$ such that the upper limit of the first term on the right-hand side is bounded by $\varepsilon/2$. By $\Pr_*\left[\widetilde{w}_{p,i}^* > 0, \, \forall i\right] \to_p 1$, the limit of the second term is zero. This gives $R_+^*(\varDelta_+^*(\tau), \tau) = O_p^*(\log(n)/\sqrt{nh})$, uniformly in $\tau \in [\underline{\tau}, \overline{\tau}]$.

By using the same arguments, we obtain a bootstrap analogue of (S.61). By this result, for all $M, \eta, \varepsilon > 0$,

$$\begin{aligned} &\Pr\left[\Pr_*\left[\inf_{\|\varDelta\|=M,\, \tau \in [\underline{\tau}, \overline{\tau}]} -\varDelta^\top R_+^*(\varDelta, \tau) < M\eta\right] > \varepsilon\right] \\ \leq \; &\Pr\left[\Pr_*\left[\sup_{\|\varDelta\|=M,\, \tau \in [\underline{\tau}, \overline{\tau}]} \left\| R_+^*(\varDelta, \tau) + D_+(\tau)\varDelta - R_+^*(0, \tau) \right\| \geq \eta\right] > \frac{\varepsilon}{2}\right] \end{aligned}$$

$$+\Pr\left[\Pr_*\left[\sup_{\tau\in[\underline{\tau},\overline{\tau}]}\|R_+^*(0,\tau)\|>M\left\{\inf_{\tau\in[\underline{\tau},\overline{\tau}]}\text{mineig}\left(D_+(\tau)\right)\right\}-2\eta\right]>\frac{\varepsilon}{2}\right]. \qquad \text{(S.85)}$$

In the proof of Lemma S.8, we obtained $(nh)^{-1/2}\sum_i W_{p,i}^*\xi_i^* = O_p^*(1)$. It follows from this result, (S.81), and Lemma S.9 that $\sup_{\tau\in[\underline{\tau},\overline{\tau}]}\|R_+^*(0,\tau)\| = O_p^*(1)$. Now, by Lemmas S.8 and S.9, for all $\eta,\varepsilon>0$, there exists an $M>0$ such that both terms on the right-hand side of (S.85) are bounded by $\varepsilon/2$ for all sufficiently large n. Therefore, for all $\eta,\varepsilon>0$, there exists an $M>0$ such that

$$\limsup_{n\uparrow\infty}\Pr\left[\Pr_*\left[\inf_{\|\varDelta\|=M,\,\tau\in[\underline{\tau},\overline{\tau}]}-\varDelta^\top R_+^*(\varDelta,\tau)<M\eta\right]>\varepsilon\right]<\varepsilon. \qquad \text{(S.86)}$$

Note that when $\widetilde{w}_{p,i}^*>0$ for all i, the bootstrap analogue of (S.64) holds. By the union bound, for all $M,\eta,\varepsilon>0$,

$$\begin{aligned}&\Pr\left[\Pr_*\left[\inf_{\|\varDelta\|\geq M,\,\tau\in[\underline{\tau},\overline{\tau}]}\|R_+^*(\varDelta,\tau)\|<\eta\right]>\varepsilon\right]\\&\qquad\leq\Pr\left[\Pr_*\left[\inf_{\|\varDelta\|=M,\,\tau\in[\underline{\tau},\overline{\tau}]}-\varDelta^\top R_+^*(\varDelta,\tau)<M\eta\right]>\frac{\varepsilon}{2}\right]+\Pr\left[\Pr_*\left[\exists i:\widetilde{w}_{p,i}^*\leq 0\right]>\frac{\varepsilon}{2}\right].\end{aligned}$$

It follows from this result, $\Pr_*\left[\widetilde{w}_{p,i}^*>0,\,\forall i\right]\to_p 1$, and (S.86) that for all $\varepsilon,\eta>0$, there exists $M>0$ such that

$$\limsup_{n\uparrow\infty}\Pr\left[\Pr_*\left[\inf_{\|\varDelta\|\geq M,\,\tau\in[\underline{\tau},\overline{\tau}]}\|R_+^*(\varDelta,\tau)\|<\eta\right]>\varepsilon\right]<\varepsilon. \qquad \text{(S.87)}$$

By the bootstrap analogue of (S.66), (S.87) and $\sup_{\tau\in[\underline{\tau},\overline{\tau}]}\|R_+^*(\varDelta_+^*(\tau),\tau)\| = o_p^*(1)$, we have $\varDelta_+^*(\tau)=O_p^*(1)$, uniformly in $\tau\in[\underline{\tau},\overline{\tau}]$.

By the bootstrap analogue of (S.67) and Lemma S.8, for all $M,\eta,\varepsilon>0$,

$$\begin{aligned}&\limsup_{n\uparrow\infty}\Pr\left[\Pr_*\left[\sup_{\tau\in[\underline{\tau},\overline{\tau}]}\left\|R_+^*(\varDelta_+^*(\tau),\tau)+D_+(\tau)\varDelta_+^*(\tau)-R_+^*(0,\tau)\right\|>\eta\right]>\varepsilon\right]\\&\qquad\qquad\leq\limsup_{n\uparrow\infty}\Pr\left[\Pr_*\left[\sup_{\tau\in[\underline{\tau},\overline{\tau}]}\left\|\varDelta_+^*(\tau)\right\|>M\right]>\frac{\varepsilon}{2}\right].\end{aligned}$$

By $\sup_{\tau\in[\underline{\tau},\overline{\tau}]}\left\|\varDelta_+^*(\tau)\right\|=O_p^*(1)$, one can choose $M>0$ such that the right-hand side is bounded by $\varepsilon/2$. Now it follows that

$$R_+^*(\varDelta_+^*(\tau),\tau)+D_+(\tau)\varDelta_+^*(\tau)-R_+^*(0,\tau)=o_p^*(1),$$

uniformly in $\tau\in[\underline{\tau},\overline{\tau}]$. The conclusion follows from this result and $\sup_{\tau\in[\underline{\tau},\overline{\tau}]}\|R_+^*(\varDelta_+^*(\tau),\tau)\| = o_p^*(1)$. ■

Proof of Theorem 3. By Lemmas S.9 and S.10, $\mathrm{e}_{p+1,1}^\top\varDelta_+^*(\tau)=\mathrm{e}_{p+1,1}^\top D_+^{-1}(\tau)R_+^*(0,\tau)+o_p^*(1)$, uniformly in $\tau \in [\underline{\tau}, \overline{\tau}]$, since $\sup_{\tau \in [\underline{\tau}, \overline{\tau}]} \left\| D_{+}^{-1}(\tau) \right\| < \infty$ by Assumption 4(iv). By (S.75) and $(nh)^{-1/2} \sum_i K_{s,p}(X_i^*/h) \xi_i^{*\top} = O_p^*(1)$, $s \in \{-,+\}$, obtained in the proof of Lemma S.8, we have $(nh)^{-1/2} \sum_i \left(W_{p,i}^* - \bar{W}_{p,i}^* \right) \xi_i^{*\top} = o_p^*(1)$. Substituting the linearization of Lemma S.9, we obtain the bootstrap analogue of (S.68). The analogue for $\mathrm{e}_{p+1,1}^{\top} \mathit{\Delta}_{-}^{*}(\tau)$ follows in the same way. Combining these with (S.84) and (S.68), the bias terms cancel, and the $o_p(1)$ remainders of (S.68) and of its counterpart for $\mathit{\Delta}_{-}(\tau)$ are $o_p^*(1)$ by Lemma S.6(i). Therefore, we have

$$\begin{aligned} \sqrt{nh} \left(\widehat{\vartheta}_p^*(\tau) - \widehat{\vartheta}_p^{eb}(\tau) \right) &= \mathbb{Z}_{p,n}^*(\tau) + o_p^*(1), \text{ where} \\ \mathbb{Z}_{p,n}^*(\tau) &\coloneqq \frac{1}{\sqrt{n}} \sum_i \left\{ \frac{\phi_i^*(\tau)}{\sqrt{h}} - \frac{1}{n} \sum_j \frac{\phi_j(\tau)}{\sqrt{h}} \right\}, \end{aligned} \tag{S.88}$$

uniformly in $\tau \in [\underline{\tau}, \overline{\tau}]$, where $\phi_i^*(\tau)$ denotes the bootstrap analogue of $\phi_i(\tau)$ defined by (S.70). Clearly, $\mathbb{Z}_{p,n}^*(\tau) = \mathbb{G}_n^* g_7(\cdot \mid \tau)$, and $\mathbb{Z}_{p,n}^*(\tau)$ is the bootstrap analogue of the leading term $\mathbb{Z}_{p,n}(\tau)$ in (S.69). By Mammen (1992, Theorem 1) and the Cramér–Wold device,

$$\left(\mathbb{Z}_{p,n}^*(\tau_1), \ldots, \mathbb{Z}_{p,n}^*(\tau_J) \right) \rightsquigarrow_* \left(\mathbb{Z}_p(\tau_1), \ldots, \mathbb{Z}_p(\tau_J) \right) \text{ in } \mathbb{R}^J, \tag{S.89}$$

for all $\tau_1 < \cdots < \tau_J$ in $[\underline{\tau}, \overline{\tau}]$.

Let $\delta \in (0,1)$. Consider a finite δ-net $\{\tau_1, \ldots, \tau_{N_\delta}\}$ of $[\underline{\tau}, \overline{\tau}]$, and for any $\tau \in [\underline{\tau}, \overline{\tau}]$, let $\pi_\delta(\tau) \in \{\tau_1, \ldots, \tau_{N_\delta}\}$ be chosen such that $|\pi_\delta(\tau) - \tau| < \delta$. Let $\mathbb{Z}_{p,n,\delta}^*(\tau) \coloneqq \mathbb{Z}_{p,n}^*(\pi_\delta(\tau))$ and $\mathbb{Z}_{p,\delta}(\tau) \coloneqq \mathbb{Z}_p(\pi_\delta(\tau))$. Then, by the triangle inequality,

$$\begin{aligned} \sup_{f \in BL_1(\ell^\infty[\underline{\tau}, \overline{\tau}])} \left| \mathrm{E}_* \left[f\left(\mathbb{Z}_{p,n}^* \right) \right] - \mathrm{E}\left[f\left(\mathbb{Z}_p \right) \right] \right| &\leq \left| \mathrm{E}_* \left[\left\| \mathbb{Z}_{p,n}^* - \mathbb{Z}_{p,n,\delta}^* \right\|_\infty \wedge 2 \right] \right| \\ &+ \sup_{f \in BL_1(\ell^\infty[\underline{\tau}, \overline{\tau}])} \left| \mathrm{E}_* \left[f\left(\mathbb{Z}_{p,n,\delta}^* \right) \right] - \mathrm{E}\left[f\left(\mathbb{Z}_{p,\delta} \right) \right] \right| + \left| \mathrm{E}\left[\left\| \mathbb{Z}_p - \mathbb{Z}_{p,\delta} \right\|_\infty \wedge 2 \right] \right|. \end{aligned} \tag{S.90}$$

By (S.89) and the continuous mapping theorem, $\mathbb{Z}_{p,n,\delta}^* \rightsquigarrow_* \mathbb{Z}_{p,\delta}$ in $\ell^\infty[\underline{\tau}, \overline{\tau}]$. Therefore, the second term on the right-hand side is $o_p(1)$. Let $\mathcal{T}_\delta \coloneqq \left\{ (\tau, \tau') \in [\underline{\tau}, \overline{\tau}]^2 : |\tau - \tau'| < \delta \right\}$. For the third term, note that

$$\left\| \mathbb{Z}_p - \mathbb{Z}_{p,\delta} \right\|_\infty \leq \sup_{(\tau, \tau') \in \mathcal{T}_\delta} \left| \mathbb{Z}_p(\tau) - \mathbb{Z}_p(\tau') \right|.$$

It follows from this result, the DCT and the fact that $\mathbb{Z}_p$ concentrates on the set of continuous functions on $[\underline{\tau}, \overline{\tau}]$ with probability one that

$$\lim_{\delta \downarrow 0} \mathrm{E}\left[\left\| \mathbb{Z}_p - \mathbb{Z}_{p,\delta} \right\|_\infty \wedge 2 \right] = 0. \tag{S.91}$$

For the first term on the right-hand side of (S.90), note that

$$\left\| \mathbb{Z}_{p,n}^* - \mathbb{Z}_{p,n,\delta}^* \right\|_\infty \leq \sup_{(\tau, \tau') \in \mathcal{T}_\delta} \left| \mathbb{Z}_{p,n}^*(\tau) - \mathbb{Z}_{p,n}^*(\tau') \right|.$$

Let $\mathscr{G}_\delta := \{g_7(\cdot \mid \tau) - g_7(\cdot \mid \tau') : (\tau, \tau') \in \mathcal{T}_\delta\}$. By Chernozhukov et al. (2014, Lemma A.6), $\mathscr{G}_\delta$ is VC-type with the envelope $2G_7$ and characteristics independent of n. Then, we have

$$\begin{aligned} \mathrm{E}_*\left[\sup_{(\tau,\tau')\in\mathcal{T}_\delta}\left|\mathbb{Z}^*_{p,n}(\tau) - \mathbb{Z}^*_{p,n}(\tau')\right|\right] &= \mathrm{E}_*\left[\|\mathbb{G}^*_n\|_{\mathscr{G}_\delta}\right], \text{ and} \\ \sup_{f\in\mathscr{G}_\delta}\mathbb{P}_n(f - \mathbb{P}_n f)^2 &\leq \sup_{(\tau,\tau')\in\mathcal{T}_\delta}\frac{1}{nh}\sum_i\left(\phi_i(\tau) - \phi_i(\tau')\right)^2. \end{aligned}$$

By (S.72) and (S.74), the right-hand side of the above inequality can be written as $\varpi^2(\delta) + \chi_n(\delta)$, where $\varpi(\delta)$ is deterministic with $\lim_{\delta\downarrow 0}\varpi(\delta) = 0$ and $\chi_n(\delta) = o_p(1)$ for all $\delta > 0$. By arguments similar to those in the proofs of Lemma S.1(i) and (iii), $\mathbb{P}_n G_7^2 = c^2 + o_p(1)$ for some $c > 0$, where G_7 is defined by (S.71). Let δ be an arbitrarily small number satisfying $2\varpi^2(\delta) \leq c^2/2$. Then, for any such fixed δ, since $\chi_n(\delta) = o_p(1)$ and $\mathbb{P}_n G_7^2 = c^2 + o_p(1)$, we have $|\chi_n(\delta)| \leq \varpi^2(\delta)$ and $c^2/2 \leq \mathbb{P}_n G_7^2 \leq 2c^2$ wpa1. By this result, Lemma A.6, and Corollary 5.1 of Chernozhukov et al. (2014) with $\sigma = \sqrt{2}\varpi(\delta)$, we have

$$\begin{aligned} \mathrm{E}_*\left[\|\mathbb{G}^*_n\|_{\mathscr{G}_\delta}\right] &= \mathrm{E}_*\left[\|\mathbb{G}^*_n\|_{\mathscr{G}_\delta}\right]\mathbb{1}\left(|\chi_n(\delta)| \leq \varpi^2(\delta), \frac{1}{2}c^2 \leq \mathbb{P}_n G_7^2 \leq 2c^2\right) + o_p(1) \\ &\lesssim \varpi(\delta)\sqrt{\log\left(\frac{C}{\varpi(\delta)}\right)} + \frac{\sqrt{\mathrm{E}_*\left[\max_i G_7^2(Y_i^*, X_i^*, Z_i^*)\right]}}{\sqrt{n}}\log\left(\frac{C}{\varpi(\delta)}\right) + o_p(1), \end{aligned} \tag{S.92}$$

where C is a positive constant that depends on c and on the VC characteristics of $\mathscr{G}_7$ defined in the proof of Theorem 2. By arguments similar to those in the proof of Lemma S.1(iv), we have

$$\frac{\sqrt{\mathrm{E}_*\left[\max_i G_7^2(Y_i^*, X_i^*, Z_i^*)\right]}}{\sqrt{n}} \leq \frac{\max_i G_7(Y_i, X_i, Z_i)}{\sqrt{n}} = o_p(1). \tag{S.93}$$

Since $\lim_{\delta\downarrow 0}\varpi(\delta)\sqrt{\log(C/\varpi(\delta))} = 0$ and (S.91), for any $\varepsilon > 0$, one can choose some sufficiently small δ such that the sum of the right-hand side of (S.92) and $\left|\mathrm{E}\left[\|\mathbb{Z}_p - \mathbb{Z}_{p,\delta}\|_\infty \wedge 2\right]\right|$ is bounded by $\varepsilon/2$ wpa1. Since the second term on the right-hand side of (S.90) is $o_p(1)$ for any fixed δ, it follows that the distance between $\mathbb{Z}^*_{p,n}$ and $\mathbb{Z}_p$ on the left-hand side of (S.90) is $o_p(1)$.

We have shown that the leading term $\mathbb{Z}^*_{p,n}$ in the linearization (S.88) converges in distribution to $\mathbb{Z}_p$, conditionally on the original data. Note that by the triangle inequality and $\mathbb{Z}^*_{p,n} \rightsquigarrow_* \mathbb{Z}_p$ in $\ell^\infty[\underline{\tau}, \overline{\tau}]$,

$$\begin{aligned} &\sup_{f\in BL_1(\ell^\infty[\underline{\tau},\overline{\tau}])}\left|\mathrm{E}_*\left[f\left(\sqrt{nh}\left(\widehat{\vartheta}^*_p(\cdot) - \widehat{\vartheta}^{eb}_p(\cdot)\right)\right)\right] - \mathrm{E}\left[f(\mathbb{Z}_p)\right]\right| \\ &\qquad\leq \left|\mathrm{E}_*\left[\sup_{\tau\in[\underline{\tau},\overline{\tau}]}\left|\sqrt{nh}\left(\widehat{\vartheta}^*_p(\tau) - \widehat{\vartheta}^{eb}_p(\tau)\right) - \mathbb{Z}^*_{p,n}(\tau)\right| \wedge 2\right]\right| + o_p(1). \end{aligned}$$

The conclusion follows from this result, (S.88) and Lemma S.6(iv). ∎

S.4 Proof of Theorem 4

We now sketch the proof of the first part of Theorem 4, the weak convergence of $\mathbb{Z}_p^{eb}(\cdot \mid h(\cdot))$. The argument extends the proof of Theorem 2 by indexing every sample object by the pair (τ, h) and establishing each intermediate bound uniformly over the rectangle $(\tau, h) \in [\underline{\tau}, \overline{\tau}] \times \mathcal{H}$, where $\mathcal{H} \coloneqq [\underline{c}h_0, \overline{c}h_0]$. Assumption 6 enters through the following fact: for g of bounded variation supported on $[-1,1]$, the function class $\{x \mapsto g(x/h) : h \in \mathcal{H}\}$ is VC-type with a constant envelope and VC characteristics depending only on g. We add the bandwidth index $h \in \mathcal{H}$ to the kernel weights and the Lagrange multiplier and show that the remainder terms in the statements of Lemmas S.1 and S.2 are uniform in $h \in \mathcal{H}$. We then add the index $h \in \mathcal{H}$ to any function class appearing in the proofs of Lemmas S.3–S.5. The proofs of Lemmas S.3–S.5 extend to uniform-in-bandwidth statements: the remainder terms in the statements of Lemmas S.3–S.5 are all uniform in $h \in \mathcal{H}$. These lead to $\mathbb{Z}_p^{eb}(\tau \mid h) = \mathbb{Z}_{p,n}(\tau \mid h) + o_p(1)$, uniformly in $(\tau, h) \in [\underline{\tau}, \overline{\tau}] \times \mathcal{H}$, where $\phi_i(\tau \mid h)$ denotes $\phi_i(\tau)$ computed at bandwidth h and $\mathbb{Z}_{p,n}(\tau \mid h)$ denotes $(nh)^{-1/2} \sum_i \phi_i(\tau \mid h)$. Now since $\{(\tau, h(\tau)) : \tau \in [\underline{\tau}, \overline{\tau}]\}$ is contained in the rectangle $[\underline{\tau}, \overline{\tau}] \times \mathcal{H}$, by taking $h = h(\tau)$, we have

$$\mathbb{Z}_p^{eb}(\tau \mid h(\tau)) = \frac{1}{\sqrt{nh(\tau)}} \sum_i \phi_i(\tau \mid h(\tau)) + o_p(1), \tag{S.94}$$

uniformly in $\tau \in [\underline{\tau}, \overline{\tau}]$.

Let $\tilde{g}_7(\cdot \mid \tau)$ denote $g_7(\cdot \mid \tau)$ with h replaced by $h(\tau)$, so that $\mathbb{G}_n \tilde{g}_7(\cdot \mid \tau)$ equals the leading term of (S.94). An envelope function proportional to $h_0^{-1/2} \mathbb{1}(|x| \leq \overline{c}h_0)\{1 + \|z - \mu_Z(x)\|\}$ satisfies the same moment bounds as G_7 with h replaced by h_0, so the Lindeberg envelope condition follows as in the proof of Theorem 2, and $\tilde{\mathscr{G}}_7 \coloneqq \{\tilde{g}_7(\cdot \mid \tau) : \tau \in [\underline{\tau}, \overline{\tau}]\}$ is VC-type with characteristics independent of n.

By the change of variables $x = h_0 u$ and the DCT, every kernel second moment that produced $\omega_{p,0}^{0,2}/\varphi$ in the corresponding calculation in the proof of Theorem 2 now converges to $\Omega_p(c(\tau), c(\tau'))/\varphi$. Hence $\mathbb{P}(\tilde{g}_7(\cdot \mid \tau)\tilde{g}_7(\cdot \mid \tau')) \to \mathscr{C}_{\star,p}^c(\tau, \tau')$.

For the equicontinuity condition (19.27) of van der Vaart (1998), we decompose $\phi(\tau \mid h(\tau)) = \phi_u(\tau \mid h(\tau)) - \phi_z(\tau \mid h(\tau))$. Then, we have

$$\begin{aligned}
&\mathrm{E}\left[\left(\frac{1}{\sqrt{h(\tau)}}\phi_u(\tau \mid h(\tau)) - \frac{1}{\sqrt{h(\tau')}}\phi_u(\tau' \mid h(\tau'))\right)^2\right] \\
&\lesssim \sum_{s \in \{+,-\}} \mathrm{E}\left[\left(\frac{\bar{W}_{s,p}(h(\tau))}{\sqrt{h(\tau)}\varphi_s(\tau)} U(\tau) - \frac{\bar{W}_{s,p}(h(\tau'))}{\sqrt{h(\tau')}\varphi_s(\tau')} U(\tau')\right)^2\right]
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \sum_{s\in\{+,-\}} \left\{ \mathrm{E}\left[\frac{\bar{W}_{s,p}^{2}\left(h\left(\tau\right)\right)}{h\left(\tau\right)\varphi_s^2\left(\tau\right)} \left(U\left(\tau\right)-U\left(\tau'\right)\right)^2 \right] + \left(\frac{1}{\varphi_s\left(\tau\right)}-\frac{1}{\varphi_s\left(\tau'\right)}\right)^2 \mathrm{E}\left[\frac{\bar{W}_{s,p}^{2}\left(h\left(\tau\right)\right)}{h\left(\tau\right)} U^2\left(\tau'\right)\right] \right. \\
&\quad \left. + \frac{1}{\varphi_s^2\left(\tau'\right)} \mathrm{E}\left[\left(\frac{\bar{W}_{s,p}\left(h\left(\tau\right)\right)}{\sqrt{h\left(\tau\right)}} - \frac{\bar{W}_{s,p}\left(h\left(\tau'\right)\right)}{\sqrt{h\left(\tau'\right)}} \right)^2 U^2\left(\tau'\right) \right] \right\}. \qquad \text{(S.95)}
\end{aligned}
$$

By similar arguments, the suprema of the first and second terms on the right-hand side of the second inequality over $|\tau-\tau'|<\delta_n$ are both $o(1)$ for every $\delta_n \downarrow 0$. For the third term, note that since $U^2\left(\tau'\right)<1$, by a change of variables

$$
\begin{aligned}
&\mathrm{E}\left[\left(\frac{\bar{W}_{+,p}\left(h\left(\tau\right)\right)}{\sqrt{h\left(\tau\right)}} - \frac{\bar{W}_{+,p}\left(h\left(\tau'\right)\right)}{\sqrt{h\left(\tau'\right)}} \right)^2 U^2\left(\tau'\right) \right] \\
&\qquad \leq \frac{1}{\varphi^2} \int_0^{\overline{x}/h_0} \left\{ c^{-1/2}\left(\tau\right) \mathcal{K}_{+,p,0}\left(\frac{u}{c\left(\tau\right)}\right) - c^{-1/2}\left(\tau'\right) \mathcal{K}_{+,p,0}\left(\frac{u}{c\left(\tau'\right)}\right) \right\}^2 f_X\left(h_0 u\right) \mathrm{d}u.
\end{aligned}
$$

Note that $\mathcal{K}_{+,p,0}\left(\cdot\right)$ is also Lipschitz on $\mathbb{R}$, under the Lipschitz condition in Assumption 6 and the bounded support condition in Assumption 2 for $K\left(\cdot\right)$. The right-hand side of the above inequality can be bounded by the product of an $O\left(1\right)$ term and $|\tau-\tau'|$, by the Lipschitz continuity of $\mathcal{K}_{+,p,0}\left(\cdot\right)$ and $c\left(\cdot\right)$. The same result holds when $+$ is replaced by $-$.

By adding and subtracting $\bar{W}_{s,p}\left(h\left(\tau\right)\right)\xi^{\top}\gamma\left(\tau'\right)/\sqrt{h\left(\tau\right)}$ for $s\in\{+,-\}$, we have

$$
\begin{aligned}
&\mathrm{E}\left[\left(\frac{1}{\sqrt{h\left(\tau\right)}}\phi_z\left(\tau \mid h\left(\tau\right)\right) - \frac{1}{\sqrt{h\left(\tau'\right)}}\phi_z\left(\tau' \mid h\left(\tau'\right)\right) \right)^2 \right] \\
&\lesssim \sum_{s\in\{+,-\}} \left\{ \mathrm{E}\left[\frac{\bar{W}_{s,p}^{2}\left(h\left(\tau\right)\right)}{h\left(\tau\right)} \left(\xi^{\top}\left(\gamma\left(\tau\right)-\gamma\left(\tau'\right)\right)\right)^2 \right] \right. \\
&\quad \left. + \mathrm{E}\left[\left(\frac{\bar{W}_{s,p}\left(h\left(\tau\right)\right)}{\sqrt{h\left(\tau\right)}} - \frac{\bar{W}_{s,p}\left(h\left(\tau'\right)\right)}{\sqrt{h\left(\tau'\right)}} \right)^2 \left(\xi^{\top}\gamma\left(\tau'\right)\right)^2 \right] \right\}.
\end{aligned}
$$

By the Cauchy–Schwarz inequality and the bounded conditional second moment of ξ (Assumption 3(iv)), the first term is bounded by the product of an $O\left(1\right)$ term and $\left\|\gamma\left(\tau\right)-\gamma\left(\tau'\right)\right\|^2$. As the third term on the right-hand side of the second inequality in (S.95), the second term admits an upper bound of the product of an $O\left(1\right)$ term and $|\tau-\tau'|$. All hypotheses of van der Vaart (1998, Theorem 19.28) are thus verified: $\mathbb{G}_n \tilde{g}_7 \rightsquigarrow \mathbb{Z}_p^c$ in $\ell^{\infty}\left[\underline{\tau},\overline{\tau}\right]$, a centered tight Gaussian process with covariance kernel $\mathscr{C}_{\star,p}^c$. This completes the proof of the first part.

The second part follows from similar uniform-in-bandwidth modifications to the proof of Theorem 3.

S.5 Proof of Theorem 5

Let $\widehat{\gamma}_p^{eb}$ denote the regression coefficient of Z_i in (28). Let

$$\widehat{\mu}_{+,p} := \arg\min_{\beta} \sum_i \widehat{w}_{p,i} K\left(\frac{X_i}{h}\right) I_i \left\{\widehat{\epsilon}_i - r_p^\top (X_i)\beta\right\}^2, \tag{S.96}$$

where $\widehat{\epsilon}_i := Y_i - Z_i^\top \widehat{\gamma}_p^{eb}$. Let $\widehat{\mu}_{-,p}$ be defined analogously. Clearly, $\widehat{\vartheta}_{\dagger,p}^{eb} = \mathrm{e}_{p+1,2}^\top (\widehat{\mu}_{+,p} - \widehat{\mu}_{-,p})$.

Let

$$W_{+,p,i}^{\dagger eb} := \mathrm{e}_{p+1,2}^\top \left(\Pi_{+,p}^{eb}\right)^{-1} K_{+,p}\left(\frac{X_i}{h}\right) \text{ and } \bar{W}_{+,p,i}^{\dagger} := \mathrm{e}_{p+1,2}^\top (\varphi \Lambda_{+,p})^{-1} K_{+,p}\left(\frac{X_i}{h}\right),$$

and let $\left(W_{-,p,i}^{\dagger eb}, \bar{W}_{-,p,i}^{\dagger}\right)$ be defined analogously. Denote $W_{p,i}^{\dagger eb} := W_{+,p,i}^{\dagger eb} - W_{-,p,i}^{\dagger eb}$ and $\bar{W}_{p,i}^{\dagger} := \bar{W}_{+,p,i}^{\dagger} - \bar{W}_{-,p,i}^{\dagger}$.

Lemma S.11. *Suppose that the assumptions in the statement of Lemma S.1 hold. The following results hold for all $k \in \mathbb{N}$. (i) If μ_A is uniformly continuous on $\mathbb{B} \setminus \{0\}$,*

$$\mathrm{E}\left[\frac{1}{h}\left(\bar{W}_{+,p}^{\dagger}\right)^k A\right] = \frac{\mu_{A,+}\omega_{p,1}^{0,k}}{\varphi^{k-1}} + o(1) \text{ and } \mathrm{E}\left[\frac{1}{h}\bar{W}_{+,p}\bar{W}_{+,p}^{\dagger} A\right] = \frac{\mu_{A,+}\varsigma_p}{\varphi} + o(1),$$

and

$$\mathrm{E}\left[\frac{1}{h}\left(\bar{W}_{-,p}^{\dagger}\right)^k A\right] = \frac{\mu_{A,-}\omega_{p,1}^{0,k}}{\varphi^{k-1}} + o(1) \text{ for even } k, \text{ and } \mathrm{E}\left[\frac{1}{h}\bar{W}_{-,p}\bar{W}_{-,p}^{\dagger} A\right] = -\frac{\mu_{A,-}\varsigma_p}{\varphi} + o(1),$$

where the sign flip follows from $\int_{-1}^{0} \mathcal{K}_{-,p,0}(t)\mathcal{K}_{-,p,1}(t)\,\mathrm{d}t = -\varsigma_p$. (ii) If μ_A is $(p+1)$-times continuously differentiable with uniformly continuous $\mu_A^{(p+1)}$ on $\mathbb{B} \setminus \{0\}$,

$$\frac{1}{h^2}\sum_i \widehat{w}_{p,i} W_{+,p,i}^{\dagger eb}\mu_A(X_i) = \mu_{A,+}^{(1)} + \frac{\mu_{A,+}^{(p+1)}}{(p+1)!}\omega_{+,p,1}^{p+1,1} h^p + o_p(h^p).$$

(iii) If for some $\delta > 0$, $\mu_{|A|^{1+\delta}}$ is bounded on $\mathbb{B} \setminus \{0\}$,

$$\frac{1}{nh}\sum_i W_{+,p,i} W_{+,p,i}^{\dagger eb} A_i - \mathrm{E}\left[\frac{1}{h}\bar{W}_{+,p}\bar{W}_{+,p}^{\dagger} A\right] = o_p(1).$$

The results in (ii) and (iii) with "+" replaced by "−" also hold.

Proof of Lemma S.11. Part (i) follows from the same arguments used to prove Lemma S.1(i).

For Part (ii), by (S.4),

$$\frac{1}{h^2}\sum_i \widehat{w}_{p,i}W^{\dagger eb}_{+,p,i}\mu_A(X_i) = \frac{1}{h^2}\sum_i \widehat{w}_{p,i}W^{\dagger eb}_{+,p,i}\left(r_p^\top(X_i)\mu_+\right) + \frac{1}{h^2}\sum_i \widehat{w}_{p,i}W^{\dagger eb}_{+,p,i}\frac{\mu_A^{(p+1)}\left(\dot{t}X_i\right)}{(p+1)!}X_i^{p+1}. \tag{S.97}$$

By the definition of $W^{\dagger eb}_{+,p,i}$, $h^{-2}\sum_i \widehat{w}_{p,i}W^{\dagger eb}_{+,p,i}\left(r_p^\top(X_i)\mu_+\right) = \mu^{(1)}_{A,+}$. Write

$$\begin{aligned}\frac{1}{h^2}\sum_i \widehat{w}_{p,i}W^{\dagger eb}_{+,p,i}\frac{\mu_A^{(p+1)}\left(\dot{t}X_i\right)}{(p+1)!}X_i^{p+1} &= \frac{1}{h}\sum_i \widehat{w}_{p,i}W^{\dagger eb}_{+,p,i}\frac{\left(\mu_A^{(p+1)}\left(\dot{t}X_i\right)-\mu_{A,+}^{(p+1)}\right)}{(p+1)!}\left(\frac{X_i}{h}\right)^{p+1}h^p \\ &+\left(\frac{\mu_{A,+}^{(p+1)}}{(p+1)!}\right)\left(\frac{1}{h}\sum_i \widehat{w}_{p,i}W^{\dagger eb}_{+,p,i}\left(\frac{X_i}{h}\right)^{p+1}\right)h^p. \end{aligned}\tag{S.98}$$

By (S.22), Lemma S.2, and arguments similar to those used in the proof of Lemma S.1(ii), one can show that $h^{-1}\sum_i \widehat{w}_{p,i}W^{\dagger eb}_{+,p,i}(X_i/h)^{p+1} = \omega^{p+1,1}_{+,p,1}+o_p(1)$ and the first term on the right-hand side of (S.98) is $o_p(h^p)$.

Part (iii) follows from (S.3), (S.23) and arguments similar to those in the proof of Lemma S.1(iii). ■

Proof of Theorem 5. By the partitioned regression argument as in CCFT,

$$\begin{aligned}&\widehat{\gamma}^{eb}_p = \\ &\left\{\frac{1}{h}\sum_i \widehat{w}_{p,i}K\left(\frac{X_i}{h}\right)Z_iZ_i^\top - \sum_{s\in\{+,-\}}\left(\sum_i \frac{1}{h}\widehat{w}_{p,i}Z_iK^\top_{s,p}\left(\frac{X_i}{h}\right)\right)\left(\Pi^{eb}_{s,p}\right)^{-1}\left(\sum_i \frac{1}{h}\widehat{w}_{p,i}K_{s,p}\left(\frac{X_i}{h}\right)Z_i^\top\right)\right\}^{-1} \\ &\times\left\{\frac{1}{h}\sum_i \widehat{w}_{p,i}K\left(\frac{X_i}{h}\right)Z_iY_i - \sum_{s\in\{+,-\}}\left(\sum_i \frac{1}{h}\widehat{w}_{p,i}Z_iK^\top_{s,p}\left(\frac{X_i}{h}\right)\right)\left(\Pi^{eb}_{s,p}\right)^{-1}\left(\sum_i \frac{1}{h}\widehat{w}_{p,i}K_{s,p}\left(\frac{X_i}{h}\right)Y_i\right)\right\}. \end{aligned}\tag{S.99}$$

By (S.22), Lemma S.1(iv) and Lemma S.2, we have $\max_i|n\widehat{w}_{p,i}-1| = o_p(1)$. By this result,

$$\frac{1}{h}\sum_i \widehat{w}_{p,i}K\left(\frac{X_i}{h}\right)Z_iZ_i^\top = \frac{1}{nh}\sum_i K\left(\frac{X_i}{h}\right)Z_iZ_i^\top + o_p(1).$$

By the von Bahr–Esseen inequality, the probability limit of the leading term on the right-hand side is $\mu_{ZZ^\top,\pm}\varphi/2$. By similar arguments, we derive the probability limits of all other terms on the right-hand side of (S.99). Then, we have $\widehat{\gamma}^{eb}_p \to_p \gamma$. Let $\epsilon_i := Y_i - Z_i^\top\gamma$. By using the first-order conditions for (S.96), we can write

$$\widehat{\vartheta}^{eb}_{\dagger,p} = \frac{1}{h^2}\sum_i \widehat{w}_{p,i}W^{\dagger eb}_{p,i}\epsilon_i - \left(\frac{1}{h^2}\sum_i \widehat{w}_{p,i}W^{\dagger eb}_{p,i}Z_i^\top\right)\left(\widehat{\gamma}^{eb}_p-\gamma\right). \tag{S.100}$$

By (S.22), Lemma S.2, and Chebyshev's inequality,

$$\frac{1}{h^2}\sum_i \widehat{w}_{p,i} K_{+,p}\left(\frac{X_i}{h}\right)\xi_i = O_p\left(\left(nh^3\right)^{-1/2}\right),$$

where $\xi_i \coloneqq Z_i - \mu_Z\left(X_i\right)$. Then, by this result, (S.3), and (S.23)

$$\left|\frac{1}{h^2}\sum_i \widehat{w}_{p,i} W^{\dagger eb}_{+,p,i}\xi_i\right| \leq \left\|\left(\Pi^{eb}_{+,p}\right)^{-1}\right\|\left\|\frac{1}{h^2}\sum_i \widehat{w}_{p,i} K_{+,p}\left(\frac{X_i}{h}\right)\xi_i\right\| = O_p\left(\left(nh^3\right)^{-1/2}\right).$$

By Lemma S.11(ii), the balancing condition $\mu^{(1)}_{Z,+} = \mu^{(1)}_{Z,-}$, and the above result, the coefficients of $\widehat{\gamma}^{eb}_p - \gamma$ in (S.100) are $O_p\left(\left(nh^3\right)^{-1/2}\right)$. It follows that the second term on the right-hand side of (S.100) is $o_p\left(\left(nh^3\right)^{-1/2}\right)$. By Lemma S.11(ii),

$$\frac{1}{h^2}\sum_i \widehat{w}_{p,i} W^{\dagger eb}_{+,p,i}\epsilon_i = \mu^{(1)}_{\epsilon,+} + \frac{\mu^{(p+1)}_{\epsilon,+}}{(p+1)!}\omega^{p+1,1}_{+,p,1} h^p + \frac{1}{h^2}\sum_i \widehat{w}_{p,i} W^{\dagger eb}_{+,p,i}\zeta_i + o_p\left(\left(nh^3\right)^{-1/2}\right),$$

where $\zeta_i \coloneqq \epsilon_i - \mu_\epsilon\left(X_i\right)$. A similar result with $+$ replaced by $-$ also holds. It now follows from the above result, (S.100) and the fact that $\mu^{(1)}_{\epsilon,+} - \mu^{(1)}_{\epsilon,-} = \vartheta_\dagger$ that

$$\widehat{\vartheta}^{eb}_{\dagger,p} = \vartheta_\dagger + \mathscr{B}_{\dagger,p} h^p + \frac{1}{h^2}\sum_i \widehat{w}_{p,i} W^{\dagger eb}_{p,i}\zeta_i + o_p\left(\left(nh^3\right)^{-1/2}\right). \tag{S.101}$$

By (S.22) and Lemma S.2,

$$\begin{aligned}
\frac{1}{h^2}\sum_i \widehat{w}_{p,i} W^{\dagger eb}_{p,i}\zeta_i &= \frac{1}{nh^2}\sum_i \left\{\iota_\varrho(1) + \iota'_\varrho(1)\left(V^\top_{p,i}\widehat{\lambda}_p\right)\right\} W^{\dagger eb}_{p,i}\zeta_i + o_p\left(\left(nh^3\right)^{-1/2}\right) \\
&= \frac{1}{nh^2}\sum_i W^{\dagger eb}_{p,i}\zeta_i - \left(\frac{1}{nh}\sum_i W^{\dagger eb}_{p,i}\zeta_i V^\top_{p,i}\right)\left(\frac{1}{nh}\sum_i V_{p,i}V^\top_{p,i}\right)^{-1}\left(\frac{1}{nh^2}\sum_i V_{p,i}\right) \\
&\quad + o_p\left(\left(nh^3\right)^{-1/2}\right).
\end{aligned} \tag{S.102}$$

By Lemma S.11(i, iii) applied coordinatewise,

$$\begin{aligned}
\frac{1}{nh}\sum_i W^{\dagger eb}_{p,i}\zeta_i V^\top_{p,i} &= \frac{1}{nh}\sum_i \left(W_{+,p,i}W^{\dagger eb}_{+,p,i} + W_{-,p,i}W^{\dagger eb}_{-,p,i}\right)\zeta_i \bar{Z}^\top_i \\
&= \left(\frac{\varsigma_p}{\varphi}\right)\left(\mu_{\zeta\bar{Z}^\top,+} - \mu_{\zeta\bar{Z}^\top,-}\right) + o_p(1).
\end{aligned}$$

By this result and (S.27),

$$\left(\frac{1}{nh}\sum_i W^{\dagger eb}_{p,i}\zeta_i V^\top_{p,i}\right)\left(\frac{1}{nh}\sum_i V_{p,i}V^\top_{p,i}\right)^{-1} = \left(\frac{\varsigma_p}{\omega^{0,2}_{p,0}}\right)\left(\mu_{\zeta\bar{Z}^\top,+} - \mu_{\zeta\bar{Z}^\top,-}\right)\mu^{-1}_{\bar{Z}\bar{Z}^\top,\pm} + o_p(1).$$

By the above result and (S.17),

$$\begin{aligned}
&\left(\frac{1}{nh}\sum_i W_{p,i}^{\dagger eb}\zeta_i V_{p,i}^\top\right)\left(\frac{1}{nh}\sum_i V_{p,i}V_{p,i}^\top\right)^{-1}\left(\frac{1}{nh^2}\sum_i V_{p,i}\right)\\
=\;&\left(\frac{\varsigma_p}{\omega_{p,0}^{0,2}}\right)\left(\mu_{\zeta\bar{Z}^\top,+}-\mu_{\zeta\bar{Z}^\top,-}\right)\mu_{\bar{Z}\bar{Z}^\top,\pm}^{-1}\left(\frac{1}{nh^2}\sum_i V_{p,i}\right)+o_p\left(\left(nh^3\right)^{-1/2}\right). \qquad \text{(S.103)}
\end{aligned}$$

Since $\mathrm{E}\left[\zeta \mid X\right]=0$ and $\bar{Z}\coloneqq\left(1,Z^\top\right)^\top$, we have $\mu_{\zeta\bar{Z}^\top,s}=\left(0,\mathrm{Cov}_s\left[Z,\epsilon\right]^\top\right)$ for $s\in\{-,+\}$. Combined with (S.16) and (S.31), the leading term equals

$$\left(\frac{\varsigma_p}{\omega_{p,0}^{0,2}}\right)\left(\mathrm{Cov}_+\left[Z,\epsilon\right]-\mathrm{Cov}_-\left[Z,\epsilon\right]\right)^\top\left(\mathrm{Var}_\pm\left[Z\right]\right)^{-1}\frac{1}{nh^2}\sum_i W_{p,i}Z_i=\frac{1}{nh^2}\sum_i W_{p,i}Z_i^\top\gamma_{\dagger,p},$$

where the last equality uses the definition of $\gamma_{\dagger,p}$. Recall the residualization $\xi_i\coloneqq Z_i-\mu_Z\left(X_i\right)$, so that $Z_i=\mu_Z(X_i)+\xi_i$. By (S.14),

$$\frac{1}{nh^2}\sum_i W_{p,i}Z_i^\top\gamma_{\dagger,p}=\frac{1}{nh^2}\sum_i W_{p,i}\xi_i^\top\gamma_{\dagger,p}+\frac{\left(\mu_{Z,+}^{(p+1)}\omega_{+,p,0}^{p+1,1}-\mu_{Z,-}^{(p+1)}\omega_{-,p,0}^{p+1,1}\right)^\top\gamma_{\dagger,p}}{(p+1)!}\cdot h^p+o_p\left(h^p\right),$$

which converts the $\mathscr{B}_{\dagger,p}$ bias into $\mathscr{B}_{\star\dagger,p}$. Then,

$$\begin{aligned}
\widehat{\vartheta}_{\dagger,p}^{eb} &= \vartheta_\dagger+\mathscr{B}_{\dagger,p}h^p+\frac{1}{nh^2}\sum_i W_{p,i}^{\dagger eb}\zeta_i-\frac{1}{nh^2}\sum_i W_{p,i}Z_i^\top\gamma_{\dagger,p}+o_p\left(\left(nh^3\right)^{-1/2}\right)\\
&= \vartheta_\dagger+\mathscr{B}_{\star\dagger,p}h^p+\frac{1}{nh^2}\sum_i\left(W_{p,i}^{\dagger eb}\zeta_i-W_{p,i}\xi_i^\top\gamma_{\dagger,p}\right)+o_p\left(\left(nh^3\right)^{-1/2}\right). \qquad \text{(S.104)}
\end{aligned}$$

By (S.3), (S.23), and Chebyshev's inequality, we have

$$\begin{aligned}
\frac{1}{\sqrt{nh}}\sum_i\left(W_{p,i}^{\dagger eb}\zeta_i-W_{p,i}\xi_i^\top\gamma_{\dagger,p}\right) &= \frac{1}{\sqrt{nh}}\sum_i\left(W_{p,i}^{\dagger}\zeta_i-W_{p,i}\xi_i^\top\gamma_{\dagger,p}\right)+o_p(1)\\
&= \frac{1}{\sqrt{nh}}\sum_i\left(\bar{W}_{p,i}^{\dagger}\zeta_i-\bar{W}_{p,i}\xi_i^\top\gamma_{\dagger,p}\right)+o_p(1).
\end{aligned}$$

By this result and (S.104),

$$\sqrt{nh^3}\left(\widehat{\vartheta}_{\dagger,p}^{eb}-\vartheta_\dagger-\mathscr{B}_{\star\dagger,p}h^p\right)=\frac{1}{\sqrt{nh}}\sum_i\left(\bar{W}_{p,i}^{\dagger}\zeta_i-\bar{W}_{p,i}\xi_i^\top\gamma_{\dagger,p}\right)+o_p(1).$$

Since $\mathrm{E}\left[\zeta \mid X\right]=0$ and $\mathrm{E}\left[\xi \mid X\right]=0$, while $\bar{W}_p^\dagger$ and $\bar{W}_p$ are functions of X alone, each summand in the preceding display has mean zero by the LIE. As the summands are i.i.d., the variance of their normalized sum equals the expectation of the squared summand divided by h, which expands as

$$\begin{aligned}\mathrm{Var}\left[\frac{1}{\sqrt{nh}}\sum_i\left(\bar{W}^{\dagger}_{p,i}\zeta_i-\bar{W}_{p,i}\xi_i^{\top}\gamma_{\dagger,p}\right)\right] &= \mathrm{E}\left[\frac{1}{h}\left(\bar{W}^{\dagger}_{p}\right)^2\zeta^2\right]-2\cdot\mathrm{E}\left[\frac{1}{h}\bar{W}^{\dagger}_{p}\bar{W}_p\zeta\xi^{\top}\gamma_{\dagger,p}\right]\\ &\quad+\mathrm{E}\left[\frac{1}{h}\bar{W}_p^2\left(\xi^{\top}\gamma_{\dagger,p}\right)^2\right].\end{aligned}$$

We evaluate the three pieces. By Lemma S.11(i) applied with $A=\zeta^2$ (so that $\mu_{\zeta^2,\pm}=\sigma^2$) on each side and summed,

$$\mathrm{E}\left[\frac{1}{h}\left(\bar{W}^{\dagger}_{p}\right)^2\zeta^2\right]=\frac{\omega^{0,2}_{p,1}\sigma^2}{\varphi}+o\left(1\right).$$

By Lemma S.1(i) applied with $A=\left(\xi^{\top}\gamma_{\dagger,p}\right)^2$,

$$\mathrm{E}\left[\frac{1}{h}\bar{W}_p^2\left(\xi^{\top}\gamma_{\dagger,p}\right)^2\right]=\frac{\omega^{0,2}_{p,0}\gamma_{\dagger,p}^{\top}\mathrm{Var}_{\pm}\left[Z\right]\gamma_{\dagger,p}}{\varphi}+o\left(1\right).$$

For the cross-moment, by Lemma S.11(i) applied with $A=\zeta\xi^{\top}\gamma_{\dagger,p}$ (so that $\mu_{A,\pm}=\mathrm{Cov}_{\pm}\left[Z,\epsilon\right]^{\top}\gamma_{\dagger,p}$),

$$\mathrm{E}\left[\frac{1}{h}\bar{W}^{\dagger}_{p}\bar{W}_p\zeta\xi^{\top}\gamma_{\dagger,p}\right]=\left(\frac{\varsigma_p}{\varphi}\right)\left(\mathrm{Cov}_{+}\left[Z,\epsilon\right]-\mathrm{Cov}_{-}\left[Z,\epsilon\right]\right)^{\top}\gamma_{\dagger,p}+o\left(1\right).$$

By the definition of $\gamma_{\dagger,p}$, $\mathrm{Cov}_{+}[Z,\epsilon]-\mathrm{Cov}_{-}[Z,\epsilon]=(\omega^{0,2}_{p,0}/\varsigma_p)\mathrm{Var}_{\pm}[Z]\gamma_{\dagger,p}$; hence

$$2\cdot\mathrm{E}\left[\frac{1}{h}\bar{W}^{\dagger}_{p}\bar{W}_p\zeta\xi^{\top}\gamma_{\dagger,p}\right]=\frac{2\omega^{0,2}_{p,0}\gamma_{\dagger,p}^{\top}\mathrm{Var}_{\pm}\left[Z\right]\gamma_{\dagger,p}}{\varphi}+o\left(1\right).$$

Collecting terms, we have

$$\mathrm{Var}\left[\frac{1}{\sqrt{nh}}\sum_i\left(\bar{W}^{\dagger}_{p,i}\zeta_i-\bar{W}_{p,i}\xi_i^{\top}\gamma_{\dagger,p}\right)\right]=\mathscr{V}_{\star\dagger,p}+o\left(1\right).\tag{S.105}$$

Now we show that $\mathscr{V}_{\star\dagger,p}$ is strictly positive. To see this, note that $\varphi\mathscr{V}_{\star\dagger,p}$ is the sum over s of the integrals of $\mathrm{Var}_s\left[\mathcal{K}_{s,p,1}\left(t\right)\epsilon-\mathcal{K}_{s,p,0}\left(t\right)Z^{\top}\gamma_{\dagger,p}\right]$ over $[0,1]$ for $s=+$ and over $[-1,0]$ for $s=-$. The integrand can be written as $a_s^{\top}\left(t\right)\mathrm{Var}_s\left[B\right]a_s\left(t\right)$ for $a_s\left(t\right):=\left(\begin{array}{cc}\mathcal{K}_{s,p,1}\left(t\right) & -\mathcal{K}_{s,p,1}\left(t\right)\gamma^{\top}-\mathcal{K}_{s,p,0}\left(t\right)\gamma_{\dagger,p}^{\top}\end{array}\right)^{\top}$, which is bounded below by $\mathrm{mineig}\left(\mathrm{Var}_s\left[B\right]\right)\mathcal{K}^2_{s,p,1}\left(t\right)$. It follows from Assumption 3(iii) that $\mathscr{V}_{\star\dagger,p}>0$. Let $\delta>0$ be the moment exponent in Assumption 3(iv). By the c_r inequality,

$$\begin{aligned}\sum_i\mathrm{E}\left[\left|\frac{1}{\sqrt{nh}}\left(\bar{W}^{\dagger}_{p,i}\zeta_i-\bar{W}_{p,i}\xi_i^{\top}\gamma_{\dagger,p}\right)\right|^{2+\delta}\right] &= \frac{1}{(nh)^{1+\delta/2}}\sum_i\mathrm{E}\left[\left|\bar{W}^{\dagger}_{p,i}\zeta_i-\bar{W}_{p,i}\xi_i^{\top}\gamma_{\dagger,p}\right|^{2+\delta}\right]\\ &\lesssim \frac{\mathrm{E}\left[h^{-1}\left|\bar{W}^{\dagger}_{p}\right|^{2+\delta}\left|\zeta\right|^{2+\delta}\right]+\mathrm{E}\left[h^{-1}\left|\bar{W}_p\right|^{2+\delta}\left|\xi^{\top}\gamma_{\dagger,p}\right|^{2+\delta}\right]}{(nh)^{\delta/2}}.\end{aligned}$$

By a change of variables and the envelope condition in Assumption 3(iv), the two expectations in the numerator are $O(1)$. By this result, (S.105) and $\mathscr{V}_{\star\dagger,p} > 0$, Lyapunov's condition

$$\frac{\sum_i \mathrm{E}\left[\left|(nh)^{-1/2}\left(\bar{W}^{\dagger}_{p,i}\zeta_i - \bar{W}_{p,i}\xi_i^{\top}\gamma_{\dagger,p}\right)\right|^{2+\delta}\right]}{\left(\mathrm{Var}\left[(nh)^{-1/2}\sum_i\left(\bar{W}^{\dagger}_{p,i}\zeta_i - \bar{W}_{p,i}\xi_i^{\top}\gamma_{\dagger,p}\right)\right]\right)^{1+\delta/2}} \to 0$$

holds. The conclusion follows from Lyapunov's central limit theorem and Slutsky's lemma. ■

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