Tuning Free Semi-parametric Estimation for First Price Auctions with Endogenous Entry

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Abstract

We propose a tuning-parameter-free method for estimating the structural parameters of semiparametric first-price auction models with endogenous entry. Our method is based on identification of the structural parameters from an infinite set of nonlinear estimating equations. We derive easy-to-verify sufficient conditions that ensure global or local identification of the copula parameter. We show that both the copula parameter and entry costs can be estimated without user-specified smoothing parameters such as bandwidths. We establish the asymptotic normality of these estimators. We further show that the entire latent value distribution can be recovered without smoothing, through a greatest-convex-minorant (Grenander-type) estimator that converges at the cube-root rate to a Chernoff-type limit. Monte Carlo simulations demonstrate that the estimators of the copula parameter, the entry costs, and the value distribution all perform well in finite samples.

Keywords: first-price auctions, endogenous entry, semiparametric estimation, tuning-parameter-free, copula, integrated quantile function
JEL Classification: C14, C57, D44

1 Introduction

Structural estimation of first-price auction models, following the seminal approach of Guerre et al. (2000) (henceforth GPV), relies on nonparametric density estimation that introduces well-known bandwidth sensitivity into the resulting estimates. The GPV method inverts the first-order condition of the bidding game, expressing the inverse bidding strategy in terms of the bid distribution $G(\cdot \mid n)$ (in auctions with n bidders) and its density $g(\cdot \mid n)$. The resulting pseudo-values form the basis for estimating the valuation distribution, and this framework has been extended to settings with risk aversion, unobserved heterogeneity, and endogenous entry (Hickman and Hubbard, 2015;

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Krasnokutskaya, 2011; Ma et al., 2019, 2021; Marmer and Shneyerov, 2012; Zincenko, 2024). A persistent practical difficulty is that kernel density estimation of $g(\cdot \mid n)$ makes the estimates sensitive to the choice of bandwidth and subject to overfitting/underfitting issues when the bandwidth is not selected optimally.

The bandwidth problem is compounded when the auction model incorporates endogenous entry. In many auction markets, potential bidders receive private signals correlated with their valuations and decide whether to incur a cost to learn their true valuations and participate. Only bidders whose signals exceed an equilibrium entry threshold choose to participate. Such a first price auction model with endogenous entry is outlined in Ye (2007). Marmer et al. (2013) and Gentry and Li (2014) develop the theoretical foundations for identification and estimation in this setting. Because signals are unobserved, when exogenous entry-cost shifters are unavailable, the joint distribution of valuations and signals cannot be identified nonparametrically (Gentry and Li, 2014). As discussed in the conclusion section of Gentry and Li (2014), a natural resolution is to adopt a semiparametric approach: model the dependence between valuations and signals through a parametric copula family $\{C(\cdot, \cdot \mid \theta) : \theta \in \Theta\}$, while leaving the marginal valuation distribution nonparametric. This imposes just enough structure for identification without sacrificing flexibility: the copula restriction, combined with the nonparametrically identified marginal distribution of valuations, is sufficient for identification of the copula parameter θ_0 and the entry costs.

This paper develops a tuning-parameter-free estimation methodology for the semiparametric endogenous entry model. Ma et al. (2025) propose a convenient method for estimating the structural parameters in this model. Ma et al. (2025) demonstrate the usefulness in an empirical study in the context of procurement auction failure. While their estimation procedure relies on the approach of GPV and requires bandwidth selection, our method eliminates this requirement entirely. Our method builds on the integrated quantile framework of Liu and Vuong (2013); Liu and Luo (2017), who apply it to test monotonicity of the inverse bidding strategy or equality of valuation distributions in standard first-price auctions without entry. Luo and Wan (2018) propose a tuning-parameter-free estimator of the valuation quantiles based on one-sided derivatives of the greatest convex minorant (GCM) of the estimated integrated valuation quantiles. Adapting this framework to the endogenous entry setting involves two key complications: the inverse bidding strategy now depends on the copula parameter through the equilibrium entry threshold, and the estimating equations must be reformulated to account for the dependence structure induced by selective entry. The key observation is that, by expressing the estimating equations as Stieltjes integrals against the empirical bid quantile function $\widehat{Q}(\cdot \mid n)$, we eliminate the need for kernel density estimation of $g(\cdot \mid n)$.

The estimator of θ_0 is defined as the minimizer of a criterion function $H(\theta)$ that compares integrated quantile representations of the latent value distribution across auction sizes. Each term in this criterion involves only the bid quantile function $Q(u \mid n)$ and known functions determined by the copula family; no density estimation is required. Because the empirical bid quantile $\widehat{Q}(\cdot \mid n)$ is a step function, each Stieltjes integral, including that defining the entry cost estimator, reduces to a finite sum over the observed bid order statistics. Because bandwidth choice is a well-known source of finite-sample sensitivity in semi-parametric/nonparametric estimation, the resulting estimators avoid this vulnerability completely.

We present four main sets of results. First, we establish identification of the copula parameter from a set of nonlinear estimating equations. Proposition 2 provides sufficient conditions for both global and local identification of θ_0. The local identification condition depends only on entry probabilities and the copula family, not on the latent valuation distribution, making it straightforward to verify empirically in practice. Second, we derive the useful representations that form the basis of our tuning-parameter-free estimators. Proposition 1 shows that the integrated quantile of (the marginal distribution of) private values can be expressed entirely in terms of the bid quantile $Q(\cdot \mid n)$ and known copula functions. Proposition 3 establishes an analogous representation for the entry cost. Third, the copula parameter estimator $\widehat{\theta}$ (Theorem 1) and the entry cost estimator (Theorem 2) are shown to be $\sqrt{L}$-asymptotically normal, where L denotes the number observed auctions. Fourth, we recover the entire latent value distribution without smoothing: the left derivative of the GCM of an integrated bid quantile yields a monotone estimator of the value quantile, and hence of the value CDF, which converges at the cube-root $L^{-1/3}$ rate to a Chernoff-type limit (Theorem 3).

We conduct Monte Carlo simulations to evaluate the finite-sample performance of our estimators. The results confirm that $\widehat{\theta}$ converges at a rate consistent with the theoretical $\sqrt{L}$-rate, exhibits near median-unbiasedness even at moderate sample sizes, and becomes approximately normally distributed as the number of auctions increases. Additional experiments confirm that the entry-cost estimators are likewise consistent at the parametric rate and approximately normal, while the greatest-convex-minorant estimator of the value distribution converges at the predicted cube-root rate. The Monte Carlo evidence demonstrates the practical merits of the tuning-free approach.

Although we develop our approach for the endogenous entry model, the underlying technique applies more broadly. Every extension of the GPV framework inherits the need to estimate the bid density $g(\cdot \mid n)$. This includes models with risk-averse bidders (Guerre et al., 2009; Campo et al., 2011), where the first-order condition involves the density ratio together with the curvature of the utility function, and models with asymmetric bidders (Flambard and Perrigne, 2006), where separate densities must be estimated for each bidder type. It also includes procurement auctions with scoring rules (Asker and Cantillon, 2008, 2010), where a change of variables reduces the equilibrium to that of a standard first-price auction. In each setting, the integrated quantile approach can in principle replace kernel density estimation with Stieltjes integration against the empirical bid quantile function. The practical impact of bandwidth sensitivity in GPV-type estimation is well documented. Henderson et al. (2012) show that structural estimates of valuation distributions vary substantially with bandwidth choice, and Hickman and Hubbard (2015) demonstrate that boundary effects in kernel density estimation distort estimates near the endpoints of the bid support. Two further considerations reinforce the case for eliminating tuning parameters altogether. First, methods that rely on tuning parameters typically suffer from overfitting or underfitting when the tuning is improper, and the optimal tuning, which depends on unknown features of the data generating process, is unknown in practice. Second, in practical implementations the tuning parameter is invariably chosen in a data-driven fashion, so that the noise in the estimated tuning parameter translates into additional noise in the estimates of the main parameters of interest. These difficulties are amplified in more complex settings. The tuning-parameter-free approach eliminates these concerns entirely.

The remainder of the paper is organized as follows. Section 2 presents the model and assumptions. Section 3 establishes identification results and derives the tuning-parameter-free representations. Section 4 introduces the estimators and presents their asymptotic properties. Section 5 develops a tuning-free estimator of the marginal value distribution and discusses counterfactual analysis. Section 6 extends the approach to auction-specific heterogeneity. Section 7 reports the Monte Carlo simulation results. All proofs are collected in the Appendix, except for certain auxiliary results that follow directly from the cited literature.

Notation. We use "$a \coloneqq b$" to denote "a is defined by b", and "$a \eqqcolon b$" is understood as "b is defined by a". $\mathbb{1}(\cdot)$ denotes the indicator function. For a positive integer T), $[T] \coloneqq \{1, ..., T\}$. The symbol $\top$ denotes transpose. For stochastic processes, convergence in distribution in the general sense (i.e., weak convergence, see Van der Vaart, 2000, Chapter 18.2) is denoted as "$\rightsquigarrow$". Convergence in distribution for random variables is denoted $\to_d$, and $\ell^\infty A$ denotes the set of bounded real-valued functions on the set A. For a finite set A, let $|A|$ denote the cardinality (number of elements).

2 Model and assumptions

Consider a first-price sealed-bid auction with $n \geq 2$ risk-neutral potential bidders for an indivisible object. We assume that n is public information. In the entry stage, the potential bidders draw their independent private signals. Each potential bidder bases her entry decision on a private signal S. A potential bidder may pay an entry cost to learn her private value V and proceed to the bidding stage, thereby becoming an active bidder. In the bidding stage, each active bidder submits a sealed bid without knowing how many other bidders entered. The object is awarded to the highest bidder, who pays her bid. The joint distribution of (V, S) is common knowledge among potential bidders. The value-signal pairs are independent across the n potential bidders.

Let $F(\cdot)$ be the marginal cumulative distribution function (CDF) of the private value V. We impose the following assumption on the marginal distribution of V.

Assumption 1 (Marginal). *$F(\cdot)$ is absolutely continuous with a continuous probability density function (PDF) $f(\cdot)$ that is compactly supported on $[\underline{v}, \overline{v}]$.*

As in Ma et al. (2025), we assume that S is uniformly distributed on $[0, 1]$.[1] Let $F_{V|S}(\cdot \mid s)$ be the conditional CDF of V given $S = s$. As in Gentry and Li (2014), we impose a parametric specification for the copula function of (V, S). Consider a one-parameter copula family $\{C(\cdot, \cdot \mid \theta) : \theta \in \Theta \subseteq \mathbb{R}\}$, and denote by θ_0 the true copula parameter, so that the joint CDF of (V, S) is given by $(v, s) \mapsto$

[1] S can be viewed as the rank of some unobserved signal. See Ma et al. (2025) for detailed discussion.

$C(F(v), s \mid \theta_0)$. For brevity, we write $C(u,s)$ for $C(u, s \mid \theta_0)$. We impose the following assumptions on the copula family.

Assumption 2 (Copula). *For each $\theta \in \Theta$, (i) $C(\cdot,\cdot \mid \theta)$ is symmetric; (ii) $C(\cdot,\cdot \mid \theta)$ is twice continuously differentiable; (iii) $C_{22}(x,y \mid \theta) := \partial^2 C(x,y \mid \theta)/\partial y^2 < 0$ for all $x,y \in [0,1]$; (iv) $C(x,y \mid \cdot)$ is continuously differentiable with $C_\theta(x,y \mid \theta) := \partial C(x,y \mid \theta)/\partial\theta \geq 0$.*

Assumption 2(iii) is the "good news " condition of Marmer et al. (2013). Assumption 2(iv) imposes positive ordering, following Ma et al. (2025). A higher value of θ corresponds to a stronger association between values and signals, as measured by concordance statistics such as Spearman's ρ. Many commonly used copula families satisfy Assumption 2, including the Gaussian copula and several Archimedean copulas (Ali-Mikhail-Haq, Clayton, Frank, Gumbel, and Joe).

Let $F^*(v \mid s) := \Pr[V \leq v \mid S \geq s]$. By the law of total probability,

$$F^*(v \mid s) = \frac{F(v) - C(F(v), s)}{1-s}. \tag{1}$$

Let

$$\begin{aligned} \Lambda(v \mid s) &:= s + (1-s) F^*(v \mid s) \\ &= F(v) + s - C(F(v), s). \end{aligned} \tag{2}$$

We consider only pure-strategy symmetric equilibria. Let $p \in [0,1]$ be the entry threshold of a typical competitor, who becomes an active bidder if her signal exceeds p. Then, in a symmetric equilibrium of the bidding stage, the winning probability of an active bidder with private value v is given by $\Lambda^{n-1}(v \mid p)$. Marmer et al. (2013) show that the equilibrium bidding strategy is

$$\beta(v \mid p, n) := v - \int_{\underline{v}}^{v} \left(\frac{\Lambda(t \mid p)}{\Lambda(v \mid p)}\right)^{n-1} \mathrm{d}t. \tag{3}$$

Define $C_2(x,y \mid \theta) := \partial C(x,y \mid \theta)/\partial y$ and $C_2(\cdot,\cdot) = C_2(\cdot,\cdot \mid \theta_0)$. The ex ante expected revenue of a typical potential bidder who observes signal s is

$$\begin{aligned} R(p,s,n) &:= \int_{\underline{v}}^{\overline{v}} (v - \beta(v \mid p, n)) \Lambda^{n-1}(v \mid p) \, \mathrm{d}F_{V|S}(v \mid s) \\ &= \int_{\underline{v}}^{\overline{v}} (1 - C_2(F(v), s)) \Lambda^{n-1}(v \mid p) \, \mathrm{d}v, \end{aligned} \tag{4}$$

where the second equality follows by the same argument as in Ma et al. (2025, Proposition 3.1). Under Assumption 2(iii), $R(p,\cdot,n)$ is strictly increasing.

Let κ_n denote the entry cost, which may vary with the number of potential bidders n. Auctions with the same n share the same entry cost κ_n. Given the entry cost κ_n, the best-response entry decision of the typical potential bidder is to enter if her signal exceeds some threshold. The equilibrium entry threshold p_n is therefore determined by

$$R(p_n, p_n, n) = \kappa_n. \tag{5}$$

We observe bids from L independent auctions. In each of these auctions, we observe equilibrium bids $\{B_{il}\}_{i=1}^{N_l^*}$ from N_l^* active bidders along with the number of potential bidders N_l. We impose the following assumption on the data-generating process (DGP).

Assumption 3 (DGP). *(i) N_l has a finite support $\mathcal{N}$ and probability masses $\{\pi_n \coloneqq \Pr[N_l = n]\}_{n \in \mathcal{N}}$. (ii) Auctions with the same number of potential bidders $N_l = n$ have the same entry cost κ_n for all $n \in \mathcal{N}$; (iii) The signals $\{S_{il}\}_{i=1}^{N_l}$ are mutually independent and independent of N_l; (iv) $N_l^* = \sum_{i=1}^{N_l} \mathbb{1}(S_{il} \geq p_{N_l})$, where p_{N_l} is the equilibrium entry threshold defined by (5); (v) The private values $\{V_{il}\}_{i=1}^{N_l^*}$ of active bidders are independent draws from the conditional distribution $F^*(\cdot \mid p_{N_l})$; (vi) The observed bids are generated by $B_{il} = \beta(V_{il} \mid p_{N_l}, N_l)$ for $i = 1, ..., N_l^*$ and $l = 1, ..., L$.*

3 Semiparametric identification

In this section, we first review the nonparametric identification results of Marmer et al. (2013). Then, we show that the copula parameter in our semiparametric specification is identifiable from a set of nonlinear estimating equations. We provide sufficient conditions for global or local identification of the parameter. Lastly, we show that under global identifiability, the copula parameter is the unique minimizer of a population objective function.

Marmer et al. (2013) show that the equilibrium entry threshold p_n is identifiable:

$$p_n = 1 - \mathrm{E}\left[\frac{N_l^*}{N_l} \mid N_l = n\right], \tag{6}$$

where the sample analogue of the right-hand side is

$$\begin{aligned} \widehat{p}_n &\coloneqq 1 - \frac{N^*(n)/n}{\sum_{l=1}^{L} \mathbb{1}(N_l = n)} \text{ where} \\ N^*(n) &\coloneqq \sum_{l: N_l = n} N_l^*. \end{aligned} \tag{7}$$

Let $G(b \mid n) \coloneqq \Pr[B_{il} \leq b \mid N_l = n]$ be the bid CDF and let $g(b \mid n) \coloneqq \partial G(b \mid n)/\partial b$ be the bid PDF. Let $\xi(\cdot \mid p, n) \coloneqq \beta^{-1}(\cdot \mid p, n)$ be the inverse bidding strategy. Marmer et al. (2013) show that

$$\xi(b \mid p_n, n) = b + \frac{G(b \mid n) + p_n/(1 - p_n)}{(n-1)\, g(b \mid n)}. \tag{8}$$

It follows from this result and (6) that $\xi(\cdot \mid p_n, n)$ is nonparametrically identifiable. It follows from

$$F^*(v \mid p_n) = \Pr\left[\xi\left(B_{il} \mid p_{N_l}, N_l\right) \leq v \mid N_l = n\right] \tag{9}$$

that $F^*(\cdot \mid p_n)$ is also nonparametrically identifiable. Let $\widehat{G}(\cdot \mid n)$ and $\widehat{g}(\cdot \mid n)$ be nonparametric estimators of $G(\cdot \mid n)$ and $g(\cdot \mid n)$. By using these estimators, (7) and (8), we can construct the plug-in estimator $\widehat{\xi}(\cdot \mid n)$ of $\xi(\cdot \mid p_n, n)$ and also the "pseudo values" $\left\{\widehat{V}_{il} \coloneqq \widehat{\xi}\left(B_{il} \mid N_l\right)\right\}_{i=1}^{N_l^*}$ for each auction.[2] Ma et al. (2025) propose using the empirical CDF $\widehat{F}^*(\cdot \mid p_n)$ of the pseudo values as an estimator of $F^*(\cdot \mid p_n)$. The density estimator $\widehat{g}(\cdot \mid n)$ of $g(\cdot \mid n)$ typically depends on smoothing (tuning) parameters such as a bandwidth.

Denote

$$\gamma(u, s \mid \theta) \coloneqq \frac{u - C(u, s \mid \theta)}{1 - s}. \tag{10}$$

By construction, $\gamma(1, s \mid \theta) = 1$ and $\gamma(0, s \mid \theta) = 0$. Since $C_1(u, s \mid \theta) \in (0, 1)$ for any $s \in (0, 1)$, we have

$$\gamma_1(u, s \mid \theta) \coloneqq \frac{\partial}{\partial u}\gamma(u, s \mid \theta) = \frac{1 - C_1(u, s \mid \theta)}{1 - s} > 0.$$

Let $\psi(\cdot, s \mid \theta)$ denote the inverse function of $\gamma(\cdot, s \mid \theta)$. Similarly, let $\psi_1(t, s \mid \theta) \coloneqq \partial\psi(t, s \mid \theta)/\partial t)$, $\psi_2(t, s \mid \theta) \coloneqq \partial\psi(t, s \mid \theta)/\partial s$ and $\gamma_2(u, s \mid \theta) \coloneqq \partial\gamma(u, s \mid \theta)/\partial s$ denote other partial derivatives.

By (1), we have

$$F^*(v \mid p_n) = \gamma\left(F(v), p_n \mid \theta_0\right), \text{ for all } (n, v) \in \mathcal{N} \times [\underline{v}, \overline{v}]. \tag{11}$$

In (11), $\{(F^*(\cdot \mid p_n), p_n)\}_{n \in \mathcal{N}}$ are known objects since they are identified. The infinitely many estimating equations in (11) can be used for identifying and estimating the unknown objects $(\theta_0, F(\cdot))$ and were proposed by Gentry and Li (2014) in their conclusion section.

Let $[v_1, ..., v_J]$ be finitely many grid points in $[\underline{v}, \overline{v}]$. We now have $|\mathcal{N}| \times J$ nonlinear estimating equations $F^*(v_j \mid p_n) = \gamma(F(v_j), p_n \mid \theta_0)$ for $(n, j) \in \mathcal{N} \times [J]$, where $[J] \coloneqq \{1, ..., J\}$, for $J + 1$ unknowns $(F(v_1), ..., F(v_J), \theta_0)$. Ma et al. (2025) propose a generalized method of moments (GMM) estimator based on $\widehat{F}^*(\cdot \mid p_n)$ and $\widehat{p}_n$ and derived the asymptotic properties. However, such an estimator depends on user-specified tuning parameters and may have undesirable finite sample performance if the tuning parameters are not chosen properly.

We now show that it is possible to avoid using tuning parameters, and propose another approach. Let $Q(\cdot)$ be the quantile function of the marginal distribution of the private values (i.e., $Q = F^{-1}$) and let $Q^*(\cdot \mid p_n)$ be the quantile function of $F^*(\cdot \mid p_n)$. By the definition of quantile functions, (11) holds if and only if

$$Q(\tau) = Q^*\left(\gamma(\tau, p_n \mid \theta_0) \mid p_n\right) \text{ for all } (n, \tau) \in \mathcal{N} \times (0, 1). \tag{12}$$

[2]See Ma et al. (2025) for details.

By (9) and the equivariance of quantiles, we have

$$\begin{aligned} Q^*(\tau \mid p_n) &= \xi(Q(\tau \mid n) \mid p_n, n) \\ &= Q(\tau \mid n) + \frac{1}{n-1} \cdot \frac{\tau + p_n/(1-p_n)}{g(Q(\tau \mid n) \mid n)}, \end{aligned} \tag{13}$$

where $Q(\tau \mid n) \coloneqq G^{-1}(\tau \mid n)$ is the bid quantile. (13) immediately gives constructive identification of $Q^*(\cdot \mid p_n)$. Moreover, (12) holds if and only if

$$\begin{aligned} I(\tau) &= \int_0^\tau Q^*(\gamma(t, p_n \mid \theta_0) \mid p_n) \, \mathrm{d}t \text{ for } (n, \tau) \in \mathcal{N} \times (0,1), \text{ where} \\ I(\tau) &\coloneqq \int_0^\tau Q(t) \, \mathrm{d}t \end{aligned} \tag{14}$$

denotes the integrated quantile.[3]

We now consider (14) as estimating equations for identifying and estimating θ_0. The key advantage of (14) over (11) is that the right-hand side of (14) with an arbitrary θ allows for tuning-parameter-free estimation. We demonstrate this result in the following proposition.

Proposition 1. *Suppose that Assumptions 1- 3 are satisfied. For all $\theta \in \Theta$,*

$$\begin{aligned} \int_0^\tau Q^*(\gamma(t, p_n \mid \theta) \mid p_n) \, \mathrm{d}t &= \int_0^{\gamma(\tau, p_n \mid \theta)} Q(u \mid n) \, \psi_1(u, p_n \mid \theta) \, \mathrm{d}u \\ &+ \frac{1}{n-1} \int_0^{\gamma(\tau, p_n \mid \theta)} \left(u + \frac{p_n}{1-p_n}\right) \psi_1(u, p_n \mid \theta) \, \mathrm{d}Q(u \mid n). \end{aligned} \tag{15}$$

When studying the global and local identification for the estimating equations (14), we consider $(\theta_0, I(\cdot))$ as the unknown parameters and treat $\{(Q^*(\cdot \mid p_n), p_n)\}_{n \in \mathcal{N}}$ as known objects since they are all identified. The following definitions follow Lewbel (2019).

Definition (Global and local identifications). (i) θ_0 is globally identified if the following condition is fulfilled. $\tilde{\theta}$ is an arbitrary point in Θ and $\tilde{I}(\cdot)$ is defined on $(0,1)$. If $\tilde{I}(\tau) = \int_0^\tau Q^*\left(\gamma\left(t, p_n \mid \tilde{\theta}\right) \mid p_n\right) \mathrm{d}t$ for all $(n, \tau) \in \mathcal{N} \times (0,1)$, then $\tilde{\theta} = \theta_0$. (ii) θ_0 is locally identified if there exists some small neighborhood around θ_0 such that the condition in (i) is satisfied with $\tilde{\theta}$ being an arbitrary point in the neighborhood.

In either case, it follows that $\tilde{I} = I$. By definition, local identification is a necessary condition for global identification. Below we provide verifiable sufficient conditions for global and local identification. As in many other cases in econometrics, the conditions for local identification are easier to verify.

[3] Let f, g be continuous functions defined on $(0,1)$. By using the fundamental theorem of calculus, we have $f(x) = g(x)$ for all $x \in (0,1)$ if and only if $\int_0^x f(t)\, dt = \int_0^x g(t)\, dt$ for all $x \in (0,1)$.

Proposition 2. *Suppose that Assumptions 1- 3 are satisfied. (i)* θ_0 *is globally identified if there exists some* $\tau \in (0,1)$ *and* $n, n' \in \mathcal{N}$ *such that*

$$\min_{\theta\in\Theta}\left\{\frac{\partial}{\partial\theta}\int_0^\tau Q^*\left(\gamma\left(t,p_n \mid \theta\right) \mid p_n\right)\mathrm{d}t-\frac{\partial}{\partial\theta}\int_0^\tau Q^*\left(\gamma\left(t,p_{n'} \mid \theta\right) \mid p_{n'}\right)\mathrm{d}t\right\}>0. \tag{16}$$

(ii) θ_0 *is locally identified if there exists some* $\tau \in (0,1)$ *and* $n, n' \in \mathcal{N}$ *such that*

$$\frac{C_\theta\left(\tau,p_n \mid \theta_0\right)}{1-C_1\left(\tau,p_n \mid \theta_0\right)}\neq\frac{C_\theta\left(\tau,p_{n'} \mid \theta_0\right)}{1-C_1\left(\tau,p_{n'} \mid \theta_0\right)}. \tag{17}$$

By Proposition 1 and Leibniz rule,

$$\begin{aligned}\frac{\partial}{\partial\theta}\int_0^\tau Q^*\left(\gamma(t,p_n \mid \theta) \mid p_n\right)dt &= Q\left(\gamma(\tau,p_n \mid \theta) \mid n\right)\psi_1\left(\gamma(\tau,p_n \mid \theta),p_n \mid \theta\right)\gamma_\theta(\tau,p_n \mid \theta)\\ &+\frac{1}{n-1}\left(\gamma(\tau,p_n \mid \theta)+\frac{p_n}{1-p_n}\right)\psi_1\left(\gamma(\tau,p_n \mid \theta),p_n \mid \theta\right)q\left(\gamma(\tau,p_n \mid \theta) \mid n\right)\gamma_\theta(\tau,p_n \mid \theta)\\ &+\int_0^{\gamma(\tau,p_n|\theta)}Q(u \mid n)\psi_{1\theta}(u,p_n \mid \theta)\mathrm{d}u+\frac{1}{n-1}\int_0^{\gamma(\tau,p_n|\theta)}\left(u+\frac{p_n}{1-p_n}\right)\psi_{1\theta}(u,p_n \mid \theta)\mathrm{d}Q(u \mid n),\end{aligned} \tag{18}$$

where $q(\cdot \mid n)$ denotes the bid quantile density, i.e., $q(u \mid n) \coloneqq 1/g\left(Q\left(u \mid n\right) \mid n\right)$, $\gamma_\theta\left(u,s \mid \theta\right) \coloneqq \partial\gamma\left(u,s \mid \theta\right)/\partial\theta$ and $\psi_{1\theta}\left(u,s \mid \theta\right) \coloneqq \partial^2\psi\left(u,s \mid \theta\right)/\partial u\partial\theta$. Expression (18) shows that empirical verification of the sufficient condition (16) for global identification requires the bid data. Moreover, nonparametric estimation of $q\left(\gamma(\tau,p_n \mid \theta) \mid n\right)$ requires using smoothing (tuning) parameters and the estimators usually converge at nonparametric rates.

In contrast, the conditions (17) for local identification are easier to verify empirically. One striking observation is that estimating terms in (17) does not even need the bid data: the estimator (7) for p_n depends only on the numbers of submitted bids. Moreover, the estimator (7) converges at a faster parametric rate.

Under Assumption 2(iv), the terms in the numerator and denominator on both sides of (17) are positive. By straightforward calculations, (17) holds if there exists some $\tau \in (0,1)$ and $n, n' \in \mathcal{N}$ such that

$$\min_{s\in[p_n\wedge p_{n'},p_n\vee p_{n'}]}C_{2\theta}\left(\tau,s \mid \theta_0\right)>0, \tag{19}$$

where $C_{2\theta}\left(u,s \mid \theta\right) \coloneqq \partial^2 C\left(u,s \mid \theta\right)/\partial u\partial\theta$. Both (17) and (19) depend only on the equilibrium entry thresholds $\{p_n\}_{n\in\mathcal{N}}$, which can be estimated from the data. Since θ_0 is unknown, either (17) or (19) must be verified for all $\theta \in \Theta$.

Given that the copula parameter and the marginal distribution of private values are identified from the estimating equations, the entry cost κ_n is immediately identified from rationalizing the identified entry threshold p_n through the "zero profit " condition (5). Next we derive a useful representation for κ_n, which allows for tuning-parameter-free estimation.

Let

$$\begin{aligned} K(u,s\mid\theta,n) &\coloneqq (1-C_2(u,s\mid\theta))(u+s-C(u,s\mid\theta))^{n-1}, \\ K_1(u,s\mid\theta,n) &\coloneqq \frac{\partial}{\partial u}K(u,s\mid\theta,n), \end{aligned}$$

and

$$k(t,s\mid\theta,n) \coloneqq K(\psi(t,s\mid\theta),s\mid\theta,n) - \frac{1}{n-1}\left(t+\frac{s}{1-s}\right)K_1(\psi(t,s\mid\theta),s\mid\theta,n)\,\psi_1(t,s\mid\theta).$$

Proposition 3. *Suppose that Assumptions 1- 3 are satisfied. The entry cost κ_n can be represented as*

$$\kappa_n = \int_0^1 k(t,p_n\mid\theta_0,n)\,\mathrm{d}Q(t\mid n).$$

Next, we show that the nonlinear restrictions from the estimating (14) imply that θ_0 minimizes some one-dimensional objective function. Let

$$\begin{aligned} \varphi(\theta\mid n,\tau) &\coloneqq \int_0^{\gamma(\tau,p_n\mid\theta)} (\tau+\rho(u,p_n\mid\theta,n))\,\mathrm{d}Q(u\mid n), \text{ where} \\ \rho(u,s\mid\theta,n) &\coloneqq \frac{1}{n-1}\left(u+\frac{s}{1-s}\right)\psi_1(u,s\mid\theta)-\psi(u,s\mid\theta). \end{aligned}$$

We define the objective function as

$$H(\theta) \coloneqq \sum_{n,n'\in\mathcal{N}:n\neq n'} \int_0^1 \left(\varphi(\theta\mid n,\tau)-\varphi\left(\theta\mid n',\tau\right)\right)^2 \mathrm{d}\tau. \tag{20}$$

In the proof of the following proposition, we show that

$$\varphi(\theta_0\mid n,\tau) = \varphi\left(\theta_0\mid n',\tau\right) \text{ for all } \left(n,n',\tau\right)\in\mathcal{N}\times\mathcal{N}\times(0,1). \tag{21}$$

Therefore, $H(\theta_0)=0$ and θ_0 minimizes $H(\theta)$ across $\theta\in\Theta$. We also show that under global identification, θ_0 is the unique minimizer. We make the following assumption.

Assumption 4 (Identification). *The copula parameter θ_0 is globally identified.*

These key properties of the objective function are summarized in the following proposition, which provides the basis for the extremum estimation method introduced in this paper.

Proposition 4. *Suppose that Assumptions 1- 4 are satisfied. $H(\theta)$ is uniquely minimized at θ_0 and $H(\theta_0)=0$.*

4 Estimators and asymptotic properties

Using the result in Proposition 4, we propose a convenient extremum estimator for the copula parameter θ_0 which is tuning-parameter-free. Then, this estimator for θ_0 and the representation in Proposition 3 lead to a tuning free estimator for the entry cost κ_n. The asymptotic properties of these estimators are characterized as $L \uparrow \infty$. In this section, we first discuss the construction of the extremum estimator for θ_0 and our estimator for κ_n. Then, we show that these estimators have the desired asymptotic normality properties.

Let

$$\begin{aligned} \widehat{G}(b \mid n) &\coloneqq \frac{1}{N^*(n)} \sum_{l:N_l=n} \sum_{i=1}^{N_l^*} \mathbb{1}(B_{il} \leq b), \text{ where} \\ N^*(n) &\coloneqq \sum_{l:N_l=n} N_l^*, \end{aligned} \tag{22}$$

where $N^*(n)$ is defined by (7). Let

$$B_{\langle 1 \rangle} \leq B_{\langle 2 \rangle} \leq \cdots \leq B_{\langle N^*(n) \rangle}$$

be the order statistics corresponding to $\left\{\{B_{il}\}_{i=1}^{N_l^*} : N_l = n\right\}$ (i.e., bids from auctions with n potential bidders). Let

$$\widehat{Q}(\tau \mid n) \coloneqq \inf\left\{y \in \mathbb{R} : \widehat{G}(y \mid n) \geq \tau\right\}$$

be the empirical quantile, for $\tau \in (0,1]$. By convention, we take $\widehat{Q}(0 \mid n) = B_{\langle 1 \rangle}$. Since $\widehat{G}(\cdot \mid n)$ is a step function, $\widehat{Q}(\cdot \mid n)$ is also a step function:

$$\widehat{Q}(\tau \mid n) = \sum_{i=1}^{N^*(n)} \mathbb{1}\left(\tau \in \left(\frac{i-1}{N^*(n)}, \frac{i}{N^*(n)}\right]\right) B_{\langle i \rangle}. \tag{23}$$

$\widehat{Q}(\cdot \mid n)$ is the empirical bid quantile based on the $N^*(n)$ bids, with $N^*(n)/L \to_p r_n$ and $r_n \coloneqq n\pi_n(1-p_n)$. The empirical bid distribution is asymptotically linear and Gaussian, as established in the Appendix. Using $\widehat{Q}(\tau \mid n)$ and $\widehat{p}_n$, we can construct a tuning-free estimator for the objective function in (20).

Now let

$$\widehat{\varphi}(\theta \mid n, \tau) \coloneqq \int_0^{\gamma(\tau, \widehat{p}_n \mid \theta)} (\tau + \rho(u, \widehat{p}_n \mid \theta, n))\, \mathrm{d}\widehat{Q}(u \mid n) \tag{24}$$

be the plug-in estimator of $\varphi(\theta \mid n, \tau)$. Since $\widehat{Q}(\cdot \mid n)$ is a step function, $\widehat{\varphi}(\theta \mid n, \tau)$ can be written as a finite sum:

$$\widehat{\varphi}(\theta \mid n, \tau) = \sum_{i=1}^{N^*(n)-1} \left(\tau + \rho\left(\frac{i}{N^*(n)}, \widehat{p}_n \mid \theta, n\right)\right) \left(B_{\langle i+1 \rangle} - B_{\langle i \rangle}\right) \mathbb{1}\left(\frac{i}{N^*(n)} < \gamma(\tau, \widehat{p}_n \mid \theta)\right).$$

Therefore, $\widehat{\varphi}(\theta \mid n, \tau)$ is straightforward to compute.

Let $\rho_1(u, s \mid \theta, n) := \partial\rho(u, s \mid \theta, n)/\partial u$ and $\rho_2(u, s \mid \theta, n) := \partial\rho(u, s \mid \theta, n)/\partial s$ be partial derivatives. Denote the following auxiliary functions (note that ϕ is distinct from the integrated quantile representation φ):

$$
\begin{aligned}
\phi(\tau, n) &:= \frac{\gamma(\tau, p_n \mid \theta_0) + p_n/(1-p_n)}{(n-1)\,\gamma_1(\tau, p_n \mid \theta_0)} q\left(\gamma(\tau, p_n \mid \theta_0) \mid n\right)\gamma_2(\tau, p_n \mid \theta_0) \\
&\quad + \int_0^{\gamma(\tau, p_n|\theta_0)} \rho_2(u, p_n \mid \theta_0, n)\mathrm{d}Q(u \mid n), \\
\pi(u \mid \tau, n) &:= \int_u^{\gamma(\tau, p_n|\theta_0)} \rho_1(t, p_n \mid \theta_0, n)\mathrm{d}Q(t \mid n) \\
&\quad - \frac{\gamma(\tau, p_n \mid \theta_0) + p_n/(1-p_n)}{(n-1)\,\gamma_1(\tau, p_n \mid \theta_0)} \cdot q\left(\gamma(\tau, p_n \mid \theta_0) \mid n\right).
\end{aligned}
$$

The following proposition establishes an asymptotic normality result for the stochastic process $\{\widehat{\varphi}(\theta_0 \mid n, \tau) : \tau \in (0,1)\}$.

Proposition 5. *Suppose that Assumptions 1- 3 are satisfied. Let $\{\mathbb{G}_j(\tau \mid n) : \tau \in (0,1)\}$, $j = 0, 1$, be independent tight Gaussian processes with covariances defined as follows:*

$$
\begin{aligned}
\mathrm{Cov}\left[\mathbb{G}_0(\tau \mid n), \mathbb{G}_0(\tau' \mid n)\right] &= \frac{1}{r_n}\Bigg\{\int_0^{\gamma(\tau\wedge\tau', p_n)} \pi(u \mid \tau, n)\,\pi(u \mid \tau', n)\,\mathrm{d}u \\
&\quad - \left(\int_0^{\gamma(\tau, p_n)} \pi(u \mid \tau, n)\,\mathrm{d}u\right)\left(\int_0^{\gamma(\tau', p_n)} \pi(u \mid \tau', n)\,\mathrm{d}u\right)\Bigg\} \quad (25)
\end{aligned}
$$

and

$$
\mathrm{Cov}\left[\mathbb{G}_1(\tau \mid n), \mathbb{G}_1(\tau' \mid n)\right] = \left(\frac{p_n(1-p_n)}{n \cdot \pi_n}\right)\phi(\tau, n)\phi(\tau', n). \quad (26)
$$

Let $\mathbb{G}(\tau \mid n) := \mathbb{G}_0(\tau \mid n) + \mathbb{G}_1(\tau \mid n)$. Then,

$$
\sqrt{L}\left(\widehat{\varphi}(\theta_0 \mid n, \cdot) - \varphi(\theta_0 \mid n, \cdot)\right) \rightsquigarrow \mathbb{G}(\cdot \mid n),
$$

in $\ell^\infty(0,1)$.

Let

$$
\widehat{H}(\theta) := \sum_{n, n' \in \mathcal{N} : n \neq n'} \int_0^1 \left(\widehat{\varphi}(\theta \mid n, \tau) - \widehat{\varphi}(\theta \mid n', \tau)\right)^2 \mathrm{d}\tau
$$

be the sample objective function. Our estimator $\widehat{\theta}$ is the approximate minimizer of $\widehat{H}(\theta)$ over $\theta \in \Theta$. This one-dimensional minimization problem can be solved by grid search. We make the following standard assumption regarding the optimality property of $\widehat{\theta}$ that we actually compute from the data.

Assumption 5 (**Approximate minimizer**). *Our estimator $\widehat{\theta}$ satisfies $\widehat{H}(\widehat{\theta}) \leq \inf_{\theta\in\Theta} \widehat{H}(\theta) + o_p\left(L^{-1}\right)$.*

The following assumption is also mild and often invoked in proofs of asymptotic normality properties of extremum estimators (see, e.g., Newey and McFadden, 1994).

Assumption 6 (Copula parameter). *Θ is compact and θ_0 is an interior point of Θ.*

Let

$$\mathit{\Gamma}\left(n, n', \tau\right) := \frac{\partial}{\partial \theta}\left(\varphi(\theta \mid n, \tau) - \varphi(\theta \mid n', \tau)\right)\Big|_{\theta=\theta_0}.$$

By slight abuse of notation, we denote $\mathit{\Gamma}(n,\tau) := \sum_{n' \in \mathcal{N}: n' \neq n} \mathit{\Gamma}(n, n', \tau)$. Let $D_0(\tau, \tau' \mid n)$ and $D_1(\tau, \tau' \mid n)$ denote the covariances in (25) and (26) respectively. Define

$$V_j(n) := \int_0^1 \int_0^1 \mathit{\Gamma}(n,\tau)\, \mathit{\Gamma}\left(n, \tau'\right) D_j\left(\tau, \tau' \mid n\right) \mathrm{d}\tau \mathrm{d}\tau',$$

for $j = 0, 1$ and $V(n) := V_0(n) + V_1(n)$. Let $\bar{\mathit{\Gamma}} := \sum_{n, n' \in \mathcal{N}: n \neq n'} \int_0^1 \mathit{\Gamma}^2(n, n', \tau)\, \mathrm{d}\tau$. The following theorem establishes the asymptotic normality of $\widehat{\theta}$.

Theorem 1. *Suppose that Assumptions 1- 6 are satisfied. The estimator $\widehat{\theta}$ for θ_0 is asymptotically normal:*

$$\begin{aligned} \sqrt{L}\left(\widehat{\theta} - \theta_0\right) &\to_d \mathrm{N}\left(0, \Sigma_\theta\right), \text{ where} \\ \Sigma_\theta &:= \left(4 \sum_{n \in \mathcal{N}} V(n)\right) / \bar{\mathit{\Gamma}}^2. \end{aligned}$$

The representation given in Proposition 3 leads to the following plug-in estimator of κ_n:

$$\widehat{\kappa}_n := \int_0^1 k\left(t, \widehat{p}_n \mid \widehat{\theta}, n\right) \mathrm{d}\widehat{Q}(t \mid n).$$

It also follows from the step function representation of $\widehat{Q}(\cdot \mid n)$ that $\widehat{\kappa}_n$ can be written as a finite sum:

$$\widehat{\kappa}_n = \sum_{i=1}^{N^*(n)-1} k\left(\frac{i}{N^*(n)}, \widehat{p}_n \mid \widehat{\theta}, n\right)\left(B_{\langle i+1\rangle} - B_{\langle i\rangle}\right).$$

This representation shows the computational convenience of the estimator.

Let $k_1(t, p \mid \theta, n) := \partial k(t, p \mid \theta, n) / \partial t$, $k_2(t, p \mid \theta, n) := \partial k(t, p \mid \theta, n) / \partial p$, and $k_\theta(t, p \mid \theta, n) := \partial k(t, p \mid \theta, n) / \partial \theta$ denote the partial derivatives. Denote

$$\omega_2(n) := \int_0^1 k_2(t, p_n \mid \theta_0, n)\, \mathrm{d}Q(t \mid n),$$

and let $\omega_\theta(n)$ be defined by the same equation with k_2 replaced by k_θ. Let

$$\mathit{\Pi}(u \mid n) := \int_u^1 k_1(t, p_n \mid \theta_0, n)\, \mathrm{d}Q(t \mid n) + 2\left(\omega_\theta(n) / \bar{\mathit{\Gamma}}\right) \int_{\psi(u, p_n)}^1 \mathit{\Gamma}(n, \tau)\, \pi(u \mid \tau, n) \mathrm{d}\tau.$$

The following theorem establishes the asymptotic normality of $\widehat{\kappa}_n$.

Theorem 2. *Suppose that Assumptions 1- 6 are satisfied. The estimator $\widehat{\kappa}_n$ for the entry cost κ_n is asymptotically normal:*

$$\sqrt{L}\left(\widehat{\kappa}_n - \kappa_n\right) \to_d \mathrm{N}\left(0, \Sigma_\kappa\right),$$

where $\Sigma_\kappa := \Sigma_1 + \Sigma_2 + \Sigma_3$,

$$\begin{aligned}
\Sigma_1 &:= \left\{ \int_0^1 \Pi^2\left(u \mid n\right) \mathrm{d}u - \left(\int_0^1 \Pi\left(u \mid n\right) \mathrm{d}u \right)^2 \right\} / r_n, \\
\Sigma_2 &:= \left(\omega_2\left(n\right) + \omega_\theta\left(n\right) \int_0^1 \Gamma\left(n, \tau\right) \phi(\tau, n) \mathrm{d}\tau \right)^2 \left\{ p_n\left(1 - p_n\right) / \left(n \cdot \pi_n\right) \right\}, \\
\Sigma_3 &:= \omega_\theta^2\left(n\right) \left(\sum_{n' \in \mathcal{N}: n' \neq n} V\left(n'\right) \right) / \bar{\Gamma}^2.
\end{aligned}$$

5 Estimation of the marginal distribution

In this section, using the estimators $\left(\widehat{\theta}, \{\widehat{\kappa}_n\}_{n \in \mathcal{N}}\right)$ from the previous section, the representation in (12), and the Grenander-type estimation method proposed by Luo and Wan (2018), we construct a tuning-free estimator for the marginal distribution of the private values.

Let $I^*\left(\tau \mid p_n\right) := \int_0^\tau Q^*\left(t \mid p_n\right) \mathrm{d}t$ be the integrated quantile function of the entrants' value quantile $Q^*\left(\cdot \mid p_n\right)$. (13) immediately implies that

$$\begin{aligned}
I^*\left(\tau \mid p_n\right) &= \int_0^\tau Q\left(t \mid n\right) \mathrm{d}t + \frac{1}{n-1} \int_0^\tau \left(t + \frac{p_n}{1 - p_n} \right) \mathrm{d}Q\left(t \mid n\right) \\
&= Q\left(\tau \mid n\right) \tau - \frac{1}{n-1} \int_0^\tau \left(\left(n-2\right) t - \frac{p_n}{1-p_n} \right) \mathrm{d}Q\left(t \mid n\right).
\end{aligned}$$

Let

$$\widehat{I}^*\left(\tau \mid p_n\right) := \widehat{Q}\left(\tau \mid n\right) \tau - \frac{1}{n-1} \int_0^\tau \left(\left(n-2\right) t - \frac{\widehat{p}_n}{1-\widehat{p}_n} \right) \mathrm{d}\widehat{Q}\left(t \mid n\right).$$

It is clear from (23) that $\tau \mapsto \widehat{Q}\left(\tau \mid n\right) \tau$ is piecewise linear and left continuous, with jumps at $\{i/N^*\left(n\right)\}_{i=1}^{N^*(n)-1}$. For the second term, we have

$$\begin{aligned}
&\int_0^\tau \left(\left(n-2\right) t - \frac{\widehat{p}_n}{1-\widehat{p}_n} \right) \mathrm{d}\widehat{Q}\left(t \mid n\right) \\
&\qquad = \sum_{i=1}^{N^*(n)-1} \left(\frac{\left(n-2\right) i}{N^*\left(n\right)} - \frac{\widehat{p}_n}{1-\widehat{p}_n} \right) \left(B_{\langle i+1 \rangle} - B_{\langle i \rangle}\right) \mathbb{1}\left(\frac{i}{N^*\left(n\right)} < \tau \right).
\end{aligned}$$

Clearly the right hand side is a piecewise constant and left continuous function of τ with jumps at $\{i/N^*\left(n\right)\}_{i=1}^{N^*(n)-1}$. Therefore, $\widehat{I}^*\left(\cdot \mid p_n\right)$ is also piecewise linear and left continuous with jumps at the same locations.

Let $\widehat{Q}^*(\cdot \mid p_n)$ be the left derivative of the GCM of $\widehat{I}^*(\cdot \mid p_n)$. And let

$$\widehat{Q}(\tau) \coloneqq \frac{1}{|\mathcal{N}|} \sum_{n \in \mathcal{N}} \widehat{Q}^* \left(\gamma \left(\tau, \widehat{p}_n \mid \widehat{\theta} \right) \mid p_n \right)$$

be a tuning-free estimator of $Q(\tau)$. We take the generalized inverse of $\widehat{Q}(\cdot)$ to construct a tuning-free estimator of $F(\cdot)$: $\widehat{F}(v) \coloneqq \sup \left\{ \tau \in (0,1) : \widehat{Q}(\tau) \leq v \right\}$.

We now establish the cube-root asymptotics of the marginal estimators. We maintain Assumptions 1–6 and add the following local regularity condition, which is the entry-model counterpart of Assumption 2 of Luo and Wan (2018).

Assumption 7 (Local regularity)**.** *The value density f is continuously differentiable and bounded away from zero on its compact support $[\underline{v}, \overline{v}]$.*

At each $u \in (0,1)$ there is a closed neighborhood in $(0,1)$ on which $g(Q(\cdot \mid n) \mid n)$ is bounded and bounded away from zero and $Q(\cdot \mid n)$ is twice continuously differentiable, so that

$$q^*(u \mid p_n) = \frac{n}{n-1} q(u \mid n) + \frac{u + p_n/(1-p_n)}{n-1} q'(u \mid n)$$

is continuous and strictly positive there, where $q(u \mid n) \coloneqq 1/g(Q(u \mid n) \mid n)$ denotes the bid quantile density and $q^*(u \mid p_n) \coloneqq \partial Q^*(u \mid p_n)/\partial u$. By Guerre et al. (2000, Proposition 1(iv)), the bid distribution inherits the smoothness of F order for order, so this condition holds whenever f is continuously differentiable. Continuity of f yields only a continuously differentiable $Q(\cdot \mid n)$, which does not suffice for the cube-root asymptotics, since the latter depends on the second derivative $q^*(\cdot \mid p_n)$.

Let $\mathbb{W}$ denote a standard two-sided Brownian motion on $\mathbb{R}$ with $\mathbb{W}(0) = 0$ and $\mathrm{Var}[\mathbb{W}(t)] = |t|$, and let

$$Z \coloneqq \arg\max_{t \in \mathbb{R}} \left\{ \mathbb{W}(t) - t^2 \right\}$$

be a random variable with the standard Chernoff distribution.[4] For a fixed $u \in (0,1)$, define the n-dependent scale constants

$$a_n(u) \coloneqq \frac{(u + p_n/(1-p_n))\, q(u \mid n)}{(n-1)\sqrt{n\pi_n(1-p_n)}} \text{ and } b_n(u) \coloneqq \frac{1}{2} q^*(u \mid p_n),$$

which are both strictly positive under Assumption 7. The building blocks $\left\{ \widehat{Q}^*(\cdot \mid p_n) \right\}_{n \in \mathcal{N}}$ are cube-root consistent with an exactly rescaled Chernoff limit. Combining them across auction sizes yields the following limiting distribution for $\widehat{Q}(\tau)$. The supporting lemmas and the proof of the theorem are collected in the Appendix.

[4] Z is symmetric about zero with $\mathrm{Var}[Z] \approx 0.26$.

Theorem 3. *Suppose that Assumptions 1–6 and 7 are satisfied. For every $\tau \in (0,1)$,*

$$\sqrt[3]{L}\left(\widehat{Q}(\tau) - Q(\tau)\right) \to_d \left\{\frac{1}{|\mathcal{N}|}\sum_{n \in \mathcal{N}} c_n(\tau)\right\} Z_n,$$
$$c_n(\tau) \coloneqq 2a_n\left(\gamma\left(\tau, p_n \mid \theta_0\right)\right)^{2/3} b_n\left(\gamma\left(\tau, p_n \mid \theta_0\right)\right)^{1/3},$$

where $\{Z_n\}_{n \in \mathcal{N}}$ are independent and identically distributed Chernoff random variables.

The limit is a weighted sum of $|\mathcal{N}|$ independent rescaled Chernoff variables, with mean 0 and variance $\left(\mathrm{Var}[Z] \sum_{n \in \mathcal{N}} c_n(\tau)^2\right) / |\mathcal{N}|^2$.

Corollary 1. *Let $v \in (\underline{v}, \overline{v})$. Under the assumptions of Theorem 3,*

$$\sqrt[3]{L}\left(\widehat{F}(v) - F(v)\right) \to_d f(v)\left\{\frac{1}{|\mathcal{N}|}\sum_{n \in \mathcal{N}} c_n(F(v))\right\} Z_n,$$

with the same independent Chernoff variables $\{Z_n\}_{n \in \mathcal{N}}$ as in Theorem 3.

The proof follows by arguments similar to those of Luo and Wan (2018) and is omitted. The point estimators are tuning-parameter-free, but the scaling constants $c_n(\tau)$ involve the bid quantile density $q(\cdot \mid n)$ and $q^*(\cdot \mid p_n)$. Estimation of these nuisance objects requires tuning.

6 Incorporating auction-specific heterogeneity

Apart from the number of potential competitors $n-1$, the bidder also observes realization x of some auction-specific characteristics $X \in \mathcal{X} \subseteq \mathbb{R}^{d_x}$. The equilibrium in this general model with auction-specific heterogeneity is characterized in Marmer et al. (2013, Section 4). We can specify a parametric form for the conditional copula of the value-signal pair given $X = x$. Then we can simply replace $\left(\widehat{p}_n, \widehat{Q}(\cdot \mid n)\right)$ in (24) by their kernel smoothed counterparts as in Marmer et al. (2013, Section 4). Then, an estimator for the conditional copula parameter can be constructed analogously. However, such an approach suffers from curse of dimensionality issue when d_x is large. In this section, we propose an augmented semiparametric model as an extension of Haile et al. (2003)'s approach to circumvent the curse of dimensionality.

Following Haile et al. (2003), we assume that the private value of bidder has a multiplicatively separable form $\eta(X) V$, where V is the idiosyncratic value that is unknown to the bidder in the entry stage, $\eta(\cdot)$ is an unknown positive function defined on $\mathcal{X}$ and V is independent of X. Haile et al. (2003) shows that in the standard first price auction model, the bidding strategy has a similar multiplicatively separable form. This observation implies that $\eta(\cdot)$ is nonparametrically identified, directly from the bid data, and one can circumvent the curse of dimensionality by using a parametric estimator of $\eta(\cdot)$ to "homogenize" the bids.[5]

[5] Also see Ma et al. (2021, Section 5) for more technical details.

After knowing the total number of potential bidders, the characteristics x and the realization of a signal S that is informative about V, the bidder may choose to pay an entry cost $\kappa(x,n)$ to learn her idiosyncratic value V and enter the bidding stage. S has a Uniform$(0,1)$ ex ante distribution and the support of V is denoted by $[\underline{v},\overline{v}]$, as in Section 2. Let $F_{S|X}(\cdot \mid x)$ denote the conditional CDF of S given $X=x$. Let $\tilde{F}^*(v \mid s,x) \coloneqq \Pr[V \leq v \mid S \geq s, X=x]$ and $\tilde{\Lambda}(v \mid s,x) \coloneqq F_{S|X}(\cdot \mid x)+\left(1-F_{S|X}(\cdot \mid x)\right)\tilde{F}^*(v \mid s,x)$. Let $p \in [0,1]$ be the entry threshold. By using the same arguments used to show (3), we can show that in a symmetric bidding equilibrium, a typical bidder submits the bid

$$\tilde{\beta}(v \mid p,n,x) \coloneqq \eta(x)v - \int_{\eta(x)\underline{v}}^{\eta(x)v} \left(\frac{\tilde{\Lambda}(t/\eta(x) \mid p,x)}{\tilde{\Lambda}(v \mid p,x)}\right)^{n-1} \mathrm{d}t$$

given realization $v \in [\underline{v},\overline{v}]$ of the idiosyncratic value V. Under the additional assumption that (V,S) is jointly independent of X, we have $\tilde{\Lambda}(v \mid s,x) = \Lambda(v \mid s)$ and by change of variables, $\tilde{\beta}(v \mid p,n,x)$ takes a separable form of

$$\tilde{\beta}(v \mid p,n,x) = \eta(x)\beta(v \mid p,n), \tag{27}$$

where $\Lambda(v \mid s)$ and $\beta(v \mid p,n)$ are defined by (2) and (3) in Section 2, respectively. Now by using 27 and the independence between (V,S) and X, it is straightforward to show that the ex ante expected revenue of a typical potential bidder with $S=s$ equals $\eta(x)R(p,s,n)$, where $R(p,s,n)$ is defined by (4) in Section 2. Now it is clear that the equilibrium entry threshold is independent of x if and only if the entry cost $\kappa(x,n)$ is of a similar separable form: $\kappa(x,n) = \eta(x)\kappa_n$, where $\{\kappa_n\}_{n\in\mathcal{N}}$ are unknown constants. Under the separable specification for the entry cost, the equilibrium entry threshold p_n is still determined by (5).

We observe data from L auctions. In each auction, we observe $\left(\{B_{il}\}_{i=1}^{N_l^*}, N_l^*, N_l\right)$ and the auction-specific characteristics X_l. We summarize our assumptions on the DGP as follows.

Assumption 8 (DGP with Heterogeneity). *(i) N_l is distributed on a finite support $\mathcal{N}$ with masses $\{\pi_n\}_{n\in\mathcal{N}}$. (ii) An auction with characteristics $X_l = x$ and number of potential bidders $N_l = n$ has entry cost $\eta(x)\kappa_n$. (iii) The signals $\{S_{il}\}_{i=1}^{N_l}$ are mutually independent and independent of (X_l, N_l); (iv) $N_l^* = \sum_{i=1}^{N_l} \mathbb{1}(S_{il} \geq p_{N_l})$, where p_{N_l} is defined by (5); (v) Given $(X_l,N_l) = (x,n)$, the idiosyncratic values $\{V_{il}\}_{i=1}^{N_l^*}$ of active bidders are independent draws from the conditional distribution $F^*(\cdot \mid p_n)$; (vi) The observed bids are generated by $B_{il} = \eta(X_l)\beta(V_{il} \mid p_{N_l}, N_l)$ for $i=1,...,N_l^*$ and $l=1,...,L$.*

Under the separability assumption in Part (vi), we can write

$$\begin{aligned}
\log(B_{il}) &= \alpha_0(N_l) + \log(\eta(X_l)) + \epsilon_{il}, \text{ where} \\
\alpha_0(N_l) &\coloneqq \mathrm{E}[\beta(V_{il} \mid p_{N_l}, N_l) \mid N_l] \\
\epsilon_{il} &\coloneqq \beta(V_{il} \mid p_{N_l}, N_l) - \alpha_0(N_l).
\end{aligned}$$

Now, since N_l is a discrete random variable, we have the following $|\mathcal{N}|$-parameter representation

$$\alpha_0(N_l) = \sum_{n \in \mathcal{N}} \alpha_0(n) \mathbb{1}(N_l = n),$$

where $\{\alpha_0(n)\}_{n \in \mathcal{N}}$ are the unknown parameters. Note that under Assumption 8(v), V_{il} is conditionally independent of X_l given N_l, and therefore, $\mathrm{E}[\epsilon_{il} \mid X_l, N_l] = 0$. Therefore, $\eta(\cdot)$ is nonparametrically identified, by the standard arguments for the partially linear model.

To circumvent the curse of dimensionality, we specify a parametric model $\{\eta(\cdot \mid \vartheta) : \vartheta \in \Theta_\eta\}$, where $\Theta_\eta \subseteq \mathbb{R}^{d_\vartheta}$ and assume that $\eta(\cdot) = \eta(\cdot \mid \vartheta_0)$ for some $\vartheta_0 \in \Theta_\eta$. E.g., $\eta(x \mid \vartheta) = \exp(x^\top \vartheta)$. Note that

$$\mathrm{E}\left[\left(\log(B_{il}) - \sum_{n \in \mathcal{N}} \alpha_0(n) \mathbb{1}(N_l = n) - \log(\eta(X_l \mid \vartheta_0))\right)^2 \mid X_l, N_l\right] \\ \leq \mathrm{E}\left[\left(\log(B_{il}) - \sum_{n \in \mathcal{N}} \alpha(n) \mathbb{1}(N_l = n) - \log(\eta(X_l \mid \vartheta))\right)^2 \mid X_l, N_l\right], \quad (28)$$

for all $\{\alpha(n)\}_{n \in \mathcal{N}}$ and ϑ, under correction specification of $\{\eta(\cdot \mid \vartheta) : \vartheta \in \Theta_\eta\}$. By law of iterated expectations and the fact that N_l^* is conditionally independent of X_l given N_l under Assumption 8(iii, iv),

$$\mathrm{E}\left[\sum_{i=1}^{N_l^*}\left(\log(B_{il}) - \sum_{n \in \mathcal{N}} \alpha(n) \mathbb{1}(N_l = n) - \log(\eta(X_l \mid \vartheta))\right)^2 \mid X_l, N_l\right] \\ = N_l(1 - p_{N_l}) \mathrm{E}\left[\left(\log(B_{il}) - \sum_{n \in \mathcal{N}} \alpha(n) \mathbb{1}(N_l = n) - \log(\eta(X_l \mid \vartheta))\right)^2 \mid X_l, N_l\right]. \quad (29)$$

By (28) and (29), we have

$$\left(\{\alpha_0(n)\}_{n \in \mathcal{N}}, \vartheta_0\right) \in \underset{\{\alpha(n)\}_{n \in \mathcal{N}}, \vartheta}{\arg\min} \mathrm{E}\left[\sum_{i=1}^{N_l^*}\left(\log(B_{il}) - \sum_{n \in \mathcal{N}} \alpha(n) \mathbb{1}(N_l = n) - \log(\eta(X_l \mid \vartheta))\right)^2\right].$$

Let $\widehat{\vartheta}$ be the estimator of ϑ_0 obtained from the regression problem

$$\min_{\{\alpha(n)\}_{n \in \mathcal{N}}, \vartheta} \sum_{l=1}^{L} \sum_{i=1}^{N_l^*}\left(\log(B_{il}) - \sum_{n \in \mathcal{N}} \alpha(n) \mathbb{1}(N_l = n) - \log(\eta(X_l \mid \vartheta))\right)^2.$$

By standard arguments, we can show that $\widehat{\vartheta}$ is $\sqrt{L}$-consistent and asymptotically normal. Now, we can calculate the homogenized bids $\left\{B_{il}/\eta\left(X_l \mid \widehat{\vartheta}\right)\right\}_{i=1}^{N_l^*}$ for $l \in [L]$, which estimate the unobserved bids $\left\{\{B_{il}/\eta(X_l)\}_{i=1}^{N_l^*} : l \in [L]\right\}$ from L hypothetical homogeneous auctions. We apply the method to the homogenized bid data to estimate the copula parameter of the joint distribution of the signal and the idiosyncratic value, and also the parameters $\{\kappa_n\}_{n\in\mathcal{N}}$.

7 Monte Carlo simulations

We evaluate the finite-sample performance of the estimator via Monte Carlo simulations. Our simulation setup follows Marmer et al. (2013). The data-generating process uses a Frank copula with true parameter $\theta_0 = 5$ to model the joint distribution of values and signals, both marginally Uniform$(0, 1)$. The number of potential bidders n is drawn uniformly from $\mathcal{N} = \{2, 3, 4, 5\}$, and entry is governed by the symmetric Bayesian–Nash equilibrium threshold p_n determined by the zero-expected-profit condition with entry cost $\kappa = 0.05$. Table 1 reports the resulting thresholds: as expected, increased competition raises p_n and reduces entry.

Table 1: Equilibrium entry thresholds

Potential bidders n	Threshold p_n	Entry probability $1 - p_n$
2	0.058	0.942
3	0.246	0.754
4	0.373	0.627
5	0.465	0.535

For each replication we draw L auctions, where $L \in \{100, 250, 500, 1000, 2000\}$, generate equilibrium bids, and compute $\widehat{\theta} = \arg\min_{\theta\in\Theta} \widehat{H}(\theta)$ by grid search over 50 equally spaced points in $\Theta = [1, 10]$. The study uses $R = 1,000$ replications per sample size. Table 2 reports the Monte Carlo results. The bias decreases from 0.617 at $L = 100$ to 0.068 at $L = 2,000$, and the RMSE decreases from 2.450 to 0.512 over the same range, confirming that $\widehat{\theta}$ is consistent. The median is close to θ_0 across all sample sizes, indicating near median-unbiasedness even in small samples. The slight positive mean bias is driven by a right tail in the sampling distribution rather than systematic over-estimation.

Table 2: Monte Carlo results for $\widehat{\theta}$ ($\theta_0 = 5$, $R = 1,000$ replications)

L	Mean	Median	Std	Bias	RMSE
100	5.617	5.224	2.372	0.617	2.450
250	5.434	5.224	1.574	0.434	1.632
500	5.190	5.041	1.050	0.190	1.067
1000	5.109	5.041	0.727	0.109	0.734
2000	5.068	5.041	0.508	0.068	0.512

Figure 1 shows the sampling distribution of $\widehat{\theta}$ at each sample size. For $L = 100$ the distribution is wide and right-skewed, a consequence of the bounded parameter space that occasionally pulls estimates toward the upper boundary of the search grid. As L increases, the distribution concentrates around θ_0 and becomes approximately normal. By $L = 2,000$, virtually all estimates fall within $[4, 6]$.

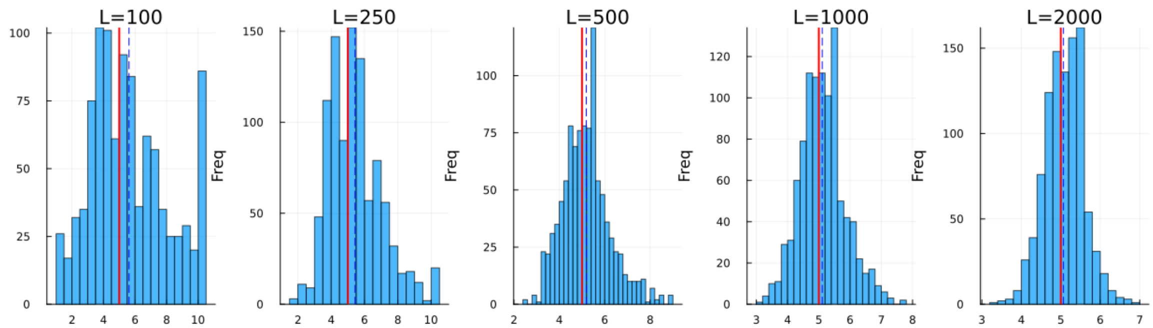

Figure 1: Histograms of $\widehat{\theta}$ across $1,000$ replications for each sample size. The solid vertical line marks $\theta_0 = 5$; the dashed line marks the sample mean.

Overall, the simulation results confirm the theoretical properties of the estimator. The key practical advantage is that the entire procedure is tuning-free: no smoothing parameter needs to be selected, which eliminates a well-known source of finite-sample sensitivity in nonparametric approaches to auction estimation.

To assess the entry-cost and marginal-distribution estimators, we report a second design in which the entry cost varies across auction sizes, with $\kappa_2, \kappa_3, \kappa_4, \kappa_5 = 0.07, 0.06, 0.05, 0.04$. The other features of the data-generating process are as above, except that the equilibrium entry thresholds adjust to the new, size-specific entry costs. Table 3 reports the entry-cost estimator $\widehat{\kappa}_n$, which plugs the copula-parameter estimator $\widehat{\theta}$ into the representation of Proposition 3, and Table 4 the GCM estimator $\widehat{Q}(\tau)$ of the marginal value quantile.

For every auction size, $\widehat{\kappa}_n$ is essentially mean-unbiased even at $L = 100$. Its mean bias never exceeds 0.002 in absolute value, although at small samples the sampling distribution is right-skewed, so the median lies somewhat below κ_n. Its root mean squared error falls at the parametric rate, roughly halving each time L quadruples, in line with the $\sqrt{L}$-asymptotic normality of Theorem 2. Precision improves with the number of bidders n, since larger auctions deliver more bids. Because the estimator substitutes $\widehat{\theta}$ for θ_0, the first-stage estimation error passes through into $\widehat{\kappa}_n$, yet the estimator remains $\sqrt{L}$-consistent and centered at the truth. Figure 2 shows the sampling distributions concentrating on the true κ_n and becoming approximately normal as L grows.

Table 4 reports the marginal value quantile estimator $\widehat{Q}(\tau)$ at $\tau \in \{0.25, 0.5, 0.75\}$. The true value is given by $Q(\tau) = \tau$.

The marginal quantile estimator carries a small downward bias that shrinks steadily with L, and its root mean squared error declines at a rate broadly consistent with the cube-root rate anticipated by Theorem 3: a log–log fit gives a rate of about -0.37, above the parametric rate $-1/2$ and close to

Table 3: Monte Carlo Results: $\widehat{\kappa}_n$ Entry-Cost Estimator (heterogeneous κ_n)

n	κ_n	L	Mean	Median	Std	Bias	RMSE
2	0.07	100	0.069	0.0596	0.0418	-0.001	0.0418
2	0.07	250	0.0681	0.065	0.0293	-0.0019	0.0294
2	0.07	500	0.0703	0.0683	0.021	0.0003	0.021
2	0.07	1000	0.0695	0.0687	0.0145	-0.0005	0.0145
2	0.07	2000	0.0695	0.069	0.0102	-0.0005	0.0102
3	0.06	100	0.0606	0.0551	0.0303	0.0006	0.0303
3	0.06	250	0.0591	0.0577	0.0209	-0.0009	0.0209
3	0.06	500	0.0601	0.0592	0.0148	0.0001	0.0148
3	0.06	1000	0.0596	0.0596	0.0102	-0.0004	0.0102
3	0.06	2000	0.0596	0.0595	0.0073	-0.0004	0.0073
4	0.05	100	0.0507	0.0477	0.0238	0.0007	0.0238
4	0.05	250	0.0496	0.0481	0.0162	-0.0004	0.0162
4	0.05	500	0.0499	0.0496	0.0112	-0.0001	0.0112
4	0.05	1000	0.0497	0.0494	0.0076	-0.0003	0.0076
4	0.05	2000	0.0498	0.0499	0.0055	-0.0002	0.0055
5	0.04	100	0.0408	0.0382	0.0193	0.0008	0.0193
5	0.04	250	0.0393	0.0387	0.0123	-0.0007	0.0123
5	0.04	500	0.0397	0.0394	0.0087	-0.0003	0.0087
5	0.04	1000	0.0398	0.0394	0.006	-0.0002	0.006
5	0.04	2000	0.04	0.04	0.0044	-0.0	0.0044

Note: $\theta_0 = 5.0$; $R = 1000$ replications.

Table 4: Monte Carlo Results: Marginal Quantile $\widehat{Q}(\tau)$ (GCM estimator) (heterogeneous κ_n)

L	τ	Mean	Median	Std	Bias	RMSE
100	0.25	0.2266	0.2227	0.0748	-0.0234	0.0783
100	0.5	0.4888	0.4837	0.0694	-0.0112	0.0702
100	0.75	0.7388	0.7376	0.058	-0.0112	0.0591
250	0.25	0.2364	0.2348	0.0522	-0.0136	0.0539
250	0.5	0.4904	0.4873	0.0492	-0.0096	0.0501
250	0.75	0.7411	0.7396	0.043	-0.0089	0.0439
500	0.25	0.243	0.2439	0.0414	-0.007	0.042
500	0.5	0.4923	0.489	0.0392	-0.0077	0.0399
500	0.75	0.7427	0.7418	0.0339	-0.0073	0.0347
1000	0.25	0.2424	0.2419	0.03	-0.0076	0.0309
1000	0.5	0.4932	0.4921	0.0287	-0.0068	0.0295
1000	0.75	0.7437	0.7429	0.0272	-0.0063	0.0279
2000	0.25	0.2438	0.2437	0.0228	-0.0062	0.0236
2000	0.5	0.4945	0.4946	0.022	-0.0055	0.0226
2000	0.75	0.7455	0.7442	0.0222	-0.0045	0.0227

Note: True $Q(\tau) = \tau$; $\theta_0 = 5.0$; (heterogeneous κ_n); $R = 1000$.

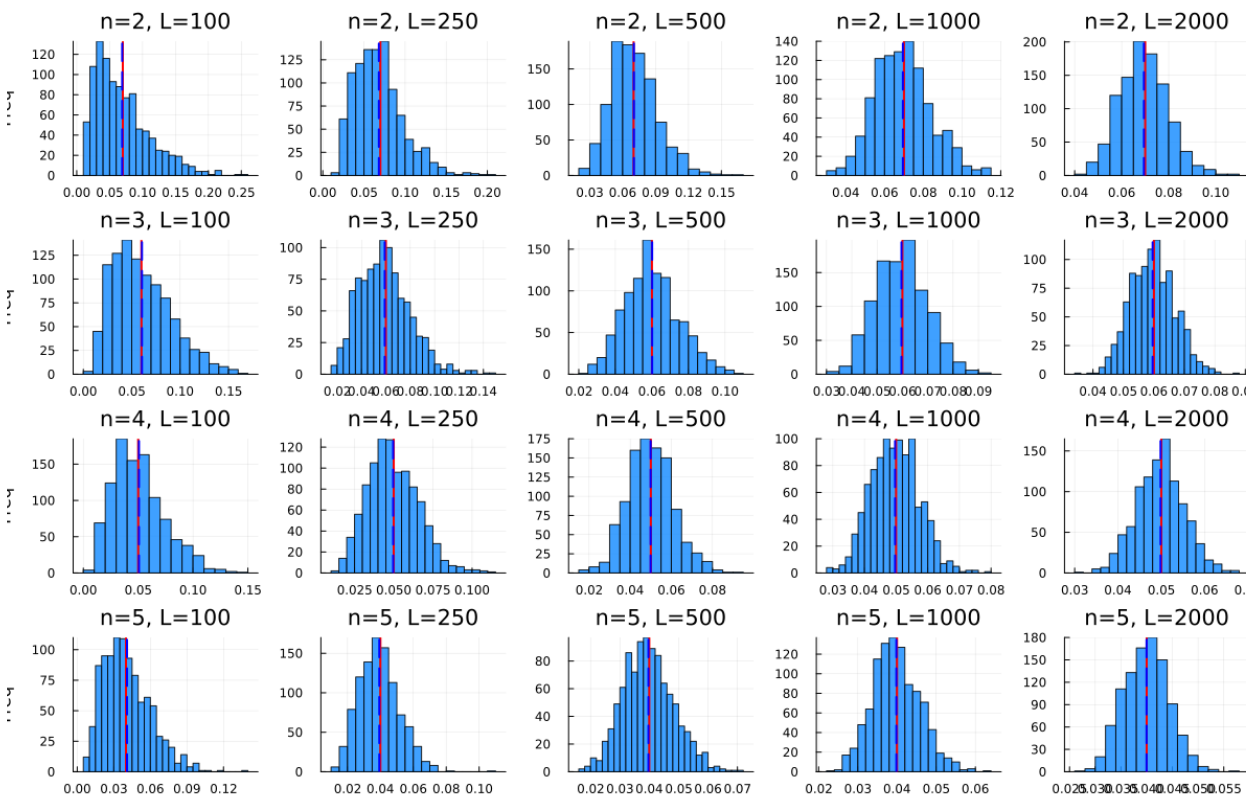

Figure 2: Sampling distribution of $\widehat{\kappa}_n$ (feasible), by auction size n (rows) and sample size L (columns); heterogeneous κ_n.

the irregular rate $-1/3$ of unsmoothed shape-constrained estimators. The estimator is well behaved throughout the interior of $[0,1]$. In unreported results, averaging each statistic over the full grid and over the trimmed grid $[0.1, 0.9]$ yields almost identical numbers, indicating that the estimates are not distorted by the boundary regions. Figure 3 displays the sampling distribution of $\widehat{Q}(0.5)$, which concentrates on the true value as L grows, consistent with the rescaled and convoluted Chernoff limit of Theorem 3.

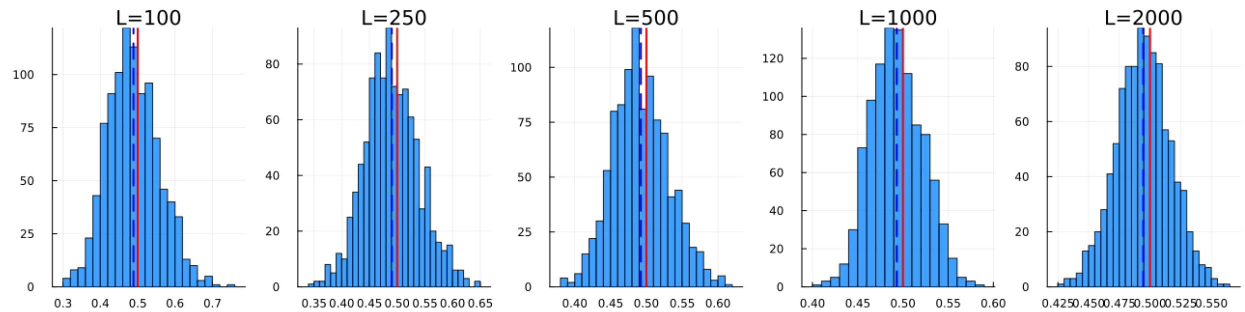

Figure 3: Sampling distribution of $\widehat{Q}(0.5)$ by sample size L; heterogeneous κ_n.

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Mathematical proofs

Appendix A Proofs of results in Section 3

Proof of Proposition 1. By (13) and change of variables $u = \gamma(t, p_n \mid \theta)$, we have

$$\begin{aligned}\int_0^\tau Q^*\left(\gamma\left(t,p_n \mid \theta\right) \mid p_n\right) \mathrm{d}t &= \int_0^\tau \left\{Q\left(\gamma\left(t,p_n \mid \theta\right) \mid n\right) + \frac{1}{n-1}\cdot\frac{\gamma\left(t,p_n \mid \theta\right) + p_n/\left(1-p_n\right)}{g\left(Q\left(\gamma\left(t,p_n \mid \theta\right) \mid n\right) \mid n\right)}\right\} \mathrm{d}t \\ &= \int_0^{\gamma(\tau,p_n\mid\theta)} \left\{Q\left(u \mid n\right) + \frac{1}{n-1}\cdot\frac{u + p_n/\left(1-p_n\right)}{g\left(Q\left(u \mid n\right) \mid n\right)}\right\} \psi_1\left(u,p_n \mid \theta\right) \mathrm{d}u \\ &= \int_0^{\gamma(\tau,p_n\mid\theta)} Q\left(u \mid n\right) \psi_1\left(u,p_n \mid \theta\right) \mathrm{d}u \\ &\quad + \frac{1}{n-1}\int_0^{\gamma(\tau,p_n\mid\theta)} \left(u + \frac{p_n}{1-p_n}\right) \frac{\psi_1\left(u,p_n \mid \theta\right)}{g\left(Q\left(u \mid n\right) \mid n\right)} \mathrm{d}u. \end{aligned} \tag{30}$$

Note that by the inverse function theorem, $q(u \mid n) = 1/g(Q(u \mid n) \mid n)$. Therefore, substituting $q(u \mid n) = 1/g(Q(u \mid n) \mid n)$ and writing the second term as a Stieltjes integral with respect to $Q(\cdot \mid n)$ yields the claim. ■

Proof of Proposition 2. Equation (14) implies that

$$\int_0^\tau Q^*\left(\gamma\left(t,p_n \mid \theta_0\right) \mid p_n\right) \mathrm{d}t = \int_0^\tau Q^*\left(\gamma\left(t,p_{n'} \mid \theta_0\right) \mid p_{n'}\right) \mathrm{d}t \tag{31}$$

for all $\tau \in (0,1)$ and $n, n' \in \mathcal{N}$. Let $\tilde{\theta}$ be a parameter value and $\tilde{I}(\cdot)$ be a function satisfying $\tilde{I}(\tau) = \int_0^\tau Q^*\left(\gamma\left(t, p_n \mid \tilde{\theta}\right) \mid p_n\right) \mathrm{d}t$ for all $(n,\tau) \in \mathcal{N} \times (0,1)$. Then, the equality (31) with θ_0 replaced by $\tilde{\theta}$ also holds, for all $\tau \in (0,1)$ and $n, n' \in \mathcal{N}$. Under (16), the function

$$\theta \mapsto \int_0^\tau Q^*\left(\gamma\left(t, p_n \mid \theta\right) \mid p_n\right) \mathrm{d}t - \int_0^\tau Q^*\left(\gamma\left(t, p_{n'} \mid \theta\right) \mid p_{n'}\right) \mathrm{d}t \tag{32}$$

must be strictly increasing and there is at most one point in Θ at which the value of the function is zero. Therefore, we must have $\tilde{\theta} = \theta_0$.

For Part (ii), note that by continuity of the derivative, (32) is locally increasing in a neighborhood around θ_0, if there exists some $\tau \in (0,1)$ and $n, n' \in \mathcal{N}$ such that

$$\frac{\partial}{\partial\theta} \int_0^\tau Q^*\big(\gamma(t, p_n \mid \theta) \mid p_n\big) \mathrm{d}t \bigg|_{\theta=\theta_0} - \frac{\partial}{\partial\theta} \int_0^\tau Q^*\big(\gamma(t, p_{n'} \mid \theta) \mid p_{n'}\big) \mathrm{d}t \bigg|_{\theta=\theta_0} > 0. \tag{33}$$

Let $q^*(u \mid p_n)$ denote $\partial Q^*(u \mid p_n)/\partial u$ and $f^*(v \mid p_n)$ denote $\partial F^*(v \mid p_n)/\partial v$. Then, we have

$$\frac{\partial}{\partial\theta} \int_0^\tau Q^*\big(\gamma(t, p_n \mid \theta) \mid p_n\big) \mathrm{d}t = \int_0^\tau q^*\big(\gamma(t, p_n \mid \theta) \mid p_n\big) \gamma_\theta(t, p_n \mid \theta) \mathrm{d}t.$$

Note that by (12), we have $\gamma(\tau, p_n \mid \theta_0) = F^*\big(Q(\tau) \mid p_n\big)$, for all $\tau \in (0,1)$ and $n \in \mathcal{N}$. Applying this result and a change of variables,

$$\frac{\partial}{\partial\theta} \int_0^\tau Q^*\big(\gamma(t, p_n \mid \theta) \mid p_n\big) \mathrm{d}t \bigg|_{\theta=\theta_0} = \int_0^{Q(\tau)} q^*\big(F^*(v \mid p_n) \mid p_n\big) \gamma_\theta(F(v), p_n \mid \theta_0) \mathrm{d}F(v). \tag{34}$$

Now by the inverse function theorem and (11),

$$q^*\big(F^*(v \mid p_n) \mid p_n\big) = \frac{1}{\gamma_1\big(F(v), p_n \mid \theta_0\big) f(v)}.$$

By these equalities, (34) and change of variables, we have

$$\frac{\partial}{\partial\theta} \int_0^\tau Q^*\big(\gamma(t, p_n \mid \theta) \mid p_n\big) \mathrm{d}t \bigg|_{\theta=\theta_0} = \int_0^\tau \frac{\gamma_\theta(t, p_n \mid \theta_0)}{\gamma_1(t, p_n \mid \theta_0)} \mathrm{d}Q(t).$$

Therefore, (33) holds if and only if there exists some $\tau \in (0,1)$ and $n, n' \in \mathcal{N}$ such that

$$\int_0^\tau \left\{ \frac{\gamma_\theta(t, p_n \mid \theta_0)}{\gamma_1(t, p_n \mid \theta_0)} - \frac{\gamma_\theta(t, p_{n'} \mid \theta_0)}{\gamma_1(t, p_{n'} \mid \theta_0)} \right\} \mathrm{d}Q(t) > 0.$$

By using the fact in Footnote 3, this condition holds if and only if there exists some $\tau \in (0,1)$ and $n, n' \in \mathcal{N}$ such that

$$\frac{\gamma_\theta(\tau, p_n \mid \theta_0)}{\gamma_1(\tau, p_n \mid \theta_0)} - \frac{\gamma_\theta(\tau, p_{n'} \mid \theta_0)}{\gamma_1(\tau, p_{n'} \mid \theta_0)} > 0.$$

Part (ii) follows from this characterization and the definition in (10). ■

Proof of Proposition 3. By (5), we have

$$\kappa_n = \int_{\underline{v}}^{\overline{v}} \left(1 - C_2\left(F\left(v\right), p_n\right)\right) \Lambda^{n-1}\left(v \mid p_n\right) \mathrm{d}v. \tag{35}$$

Note that we have

$$K\left(1, s \mid \theta, n\right) = 0 \text{ and } K\left(0, s \mid \theta, n\right) = s^{n-1}, \tag{36}$$

for all (s, θ). By (35), change of variables and integration by parts,

$$\begin{aligned} \kappa_n &= \int_0^1 K\left(u, p_n \mid \theta_0, n\right) \mathrm{d}Q\left(u\right) \\ &= -\underline{v} p_n^{n-1} - \int_0^1 Q\left(u\right) K_1\left(u, p_n \mid \theta_0, n\right) \mathrm{d}u. \end{aligned}$$

Then, by (12) and change of variables,

$$\begin{aligned} \kappa_n &= -\underline{v} p_n^{n-1} - \int_0^1 Q^*\left(\gamma\left(u, p_n \mid \theta_0\right) \mid p_n\right) K_1\left(u, p_n \mid \theta_0, n\right) \mathrm{d}u \\ &= -\underline{v} p_n^{n-1} - \int_0^1 Q^*\left(t \mid p_n\right) K_1\left(\psi\left(t, p_n \mid \theta_0\right), p_n \mid \theta_0, n\right) \psi_1\left(t, p_n \mid \theta_0\right) \mathrm{d}t. \end{aligned}$$

Combining this result with (13),

$$\begin{aligned} \kappa_n &= -\underline{v} p_n^{n-1} - \int_0^1 Q\left(t \mid n\right) K_1\left(\psi\left(t, p_n \mid \theta_0\right), p_n \mid \theta_0, n\right) \psi_1\left(t, p_n \mid \theta_0\right) \mathrm{d}t \\ &\quad - \frac{1}{n-1} \int_0^1 \left(t + \frac{p_n}{1 - p_n}\right) K_1\left(\psi\left(t, p_n \mid \theta_0\right), p_n \mid \theta_0, n\right) \psi_1\left(t, p_n \mid \theta_0\right) \mathrm{d}Q\left(t \mid n\right). \end{aligned} \tag{37}$$

By integration by parts, (36), $\psi\left(1, p_n \mid \theta_0\right) = 1$ and $\psi\left(0, p_n \mid \theta_0\right) = 0$, we have

$$\begin{aligned} & \int_0^1 Q\left(t \mid n\right) K_1\left(\psi\left(t, p_n \mid \theta_0\right), p_n \mid \theta_0, n\right) \psi_1\left(t, p_n \mid \theta_0\right) \mathrm{d}t \\ =\; & Q\left(1 \mid n\right) K\left(\psi\left(1, p_n \mid \theta_0\right), p_n \mid \theta_0, n\right) - Q\left(0 \mid n\right) K\left(\psi\left(0, p_n \mid \theta_0\right), p_n \mid \theta_0, n\right) \\ & - \int_0^1 K\left(\psi\left(t, p_n \mid \theta_0\right), p_n \mid \theta_0, n\right) \mathrm{d}Q\left(t \mid n\right) \\ =\; & -\underline{v} p_n^{n-1} - \int_0^1 K\left(\psi\left(t, p_n \mid \theta_0\right), p_n \mid \theta_0, n\right) \mathrm{d}Q\left(t \mid n\right). \end{aligned}$$

Combining these calculations with (37) yields the claim. ■

Proof of Proposition 4. By integration by parts, we have

$$\int_0^{\gamma(\tau, p_n \mid \theta)} Q\left(u \mid n\right) \psi_1\left(u, p_n \mid \theta\right) \mathrm{d}u = Q\left(\gamma\left(\tau, p_n \mid \theta\right) \mid n\right) \tau - \int_0^{\gamma(\tau, p_n \mid \theta)} \psi\left(u, p_n \mid \theta\right) \mathrm{d}Q\left(u \mid n\right).$$

By this equality and the simple fact that

$$\tau \int_0^{\gamma(\tau,p_n|\theta)} \mathrm{d}Q\left(u \mid n\right) = \tau Q\left(\gamma\left(\tau, p_n \mid \theta\right) \mid n\right) - \tau \cdot Q\left(0 \mid n\right),$$

we now have

$$\int_0^{\gamma(\tau,p_n|\theta)} Q\left(u \mid n\right) \psi_1\left(u, p_n \mid \theta\right) \mathrm{d}u = \int_0^{\gamma(\tau,p_n|\theta)} \left\{\tau - \psi\left(u, p_n \mid \theta\right)\right\} \mathrm{d}Q\left(u \mid n\right) + \tau \cdot Q\left(0 \mid n\right).$$

By this result and Proposition 1,

$$\int_0^{\tau} Q^*\left(\gamma\left(t, p_n \mid \theta\right) \mid p_n\right) \mathrm{d}t = \varphi\left(\theta \mid n, \tau\right) + \tau \cdot Q\left(0 \mid n\right). \tag{38}$$

Note that on the left-hand side of (38), $Q\left(0 \mid n\right) = \underline{v}$ for all $n \in \mathcal{N}$. Now (21) follows from these results and (14) and we have $H(\theta_0) = 0$. By (38),

$$H(\theta) = \sum_{n,n' \in \mathcal{N}: n \neq n'} \int_0^1 \left(\int_0^{\tau} Q^*\left(\gamma\left(t, p_n \mid \theta\right) \mid p_n\right) \mathrm{d}t - \int_0^{\tau} Q^*\left(\gamma\left(t, p_{n'} \mid \theta\right) \mid p_{n'}\right) \mathrm{d}t \right)^2 \mathrm{d}\tau,$$

for all $\theta \in \Theta$. If the global identification condition holds and $H(\tilde{\theta}) = 0$, we must have $\tilde{\theta} = \theta_0$. ■

Appendix B Proofs of results in Section 4

The following lemma shows the asymptotic linearization and normality of $\widehat{p}_n$.

Lemma 1. *Under the assumptions in the statement of Proposition 5, $\widehat{p}_n$ is asymptotically linear and normal:*

$$\widehat{p}_n = p_n - \frac{1}{\pi_n} \cdot \frac{1}{L} \sum_{l: N_l = n} \left\{ \frac{N_l^*}{n} - (1 - p_n) \right\} + O_p\left(L^{-1}\right), \tag{39}$$

and

$$\sqrt{L}\left(\widehat{p}_n - p_n\right) \to_d \mathrm{N}\left(0, \frac{p_n\left(1 - p_n\right)}{n \cdot \pi_n}\right).$$

Proof of Lemma 1. We write

$$\widehat{p}_n = 1 - \frac{\frac{1}{L}\sum_{l=1}^L \mathbb{1}(N_l = n)\left(N_l^*/n\right)}{\frac{1}{L}\sum_{l=1}^L \mathbb{1}(N_l = n)}.$$

Then, we have

$$\widehat{p}_n - p_n \quad = \quad -\frac{1}{\widehat{\pi}_n} \cdot \frac{1}{L} \sum_{l=1}^{L} \mathbb{1}(N_l = n) \left\{ \frac{N_l^*}{n} - (1 - p_n) \right\}, \text{ where}$$

$$\widehat{\pi}_n \quad \coloneqq \quad \frac{1}{L}\sum_{l=1}^{L}\mathbb{1}(N_l = n).$$

Note that by standard arguments, $\widehat{\pi}_n - \pi_n = O_p\left(L^{-1/2}\right)$. Since $p_n = 1 - \mathrm{E}\left[N_l^*/n \mid N_l = n\right]$, and the conditional distribution of N_l^* given $N_l = n$ is Binomial$(1-p_n, n)$, we have

$$\mathrm{E}\left[\left(\frac{N_l^*}{n} - (1-p_n)\right)^2 \mid N_l = n\right] = \frac{p_n(1-p_n)}{n}.$$

Therefore, by the central limit theorem, we have

$$\frac{1}{\sqrt{L}}\sum_{l=1}^{L}\mathbb{1}(N_l = n)\left\{\frac{N_l^*}{n} - (1-p_n)\right\} \to_d \mathrm{N}\left(0, \frac{\pi_n p_n\left(1-p_n\right)}{n}\right). \tag{40}$$

Combining this limit with $\widehat{\pi}_n - \pi_n = O_p\left(L^{-1/2}\right)$, we have (39). Asymptotic normality then follows from this result and (40). ■

Let $r_n \coloneqq \mathrm{E}\left[\mathbb{1}(N_l = n)N_l^*\right] = n\pi_n\left(1-p_n\right)$. The following lemma shows the asymptotic linearization and Gaussianity of $\widehat{G}\left(\cdot \mid n\right)$.

Lemma 2. *Under the assumptions in the statement of Proposition 5, $\widehat{G}\left(\cdot \mid n\right)$ is asymptotically linear and Gaussian:*

$$\widehat{G}\left(b \mid n\right) = G\left(b \mid n\right) + \frac{1}{r_n}\cdot\frac{1}{L}\sum_{l:N_l=n}\sum_{i=1}^{N_l^*}\left\{\mathbb{1}\left(B_{il} \le b\right) - G\left(b \mid n\right)\right\} + O_p\left(L^{-1}\right) \tag{41}$$

uniformly in $b \in \mathbb{R}$ and

$$\sqrt{L}\left(\widehat{G}\left(\cdot \mid n\right) - G\left(\cdot \mid n\right)\right) \rightsquigarrow \mathbb{B}\left(G\left(\cdot \mid n\right)\right)/\sqrt{r_n},$$

where $\{\mathbb{B}\left(t\right) : t \in [0,1]\}$ is a standard Brownian bridge.

Proof of Lemma 2. Let $\widehat{r}_n \coloneqq L^{-1}\sum_{l:N_l=n} N_l^*$. By the definition in (22), we write

$$\widehat{G}\left(b \mid n\right) - G\left(b \mid n\right) = \frac{1}{\widehat{r}_n}\cdot\frac{1}{L}\sum_{l:N_l=n}\sum_{i=1}^{N_l^*}\left(\mathbb{1}\left(B_{il} \le b\right) - G\left(b \mid n\right)\right). \tag{42}$$

Let $(B_{1l}, B_{2l}, ..., B_{nl})$ be bids from n (hypothetical) bidders who draw their values from $F^*\left(\cdot \mid p_n\right)$. Let $A_l \coloneqq \left(B_{1l}, ..., B_{nl}, N_l^*, N_l\right)^\top$ and denote

$$\zeta\left(A_l \mid b\right) \coloneqq \mathbb{1}\left(N_l = n\right)\sum_{i=1}^{N_l^*}\left\{\mathbb{1}\left(B_{il} \le b\right) - G\left(b \mid n\right)\right\}.$$

Then by using these notations, we can write

$$\frac{1}{L}\sum_{l:N_l=n}\sum_{i=1}^{N_l^*}\left(\mathbb{1}\left(B_{il}\le b\right)-G\left(b\mid n\right)\right)=\frac{1}{L}\sum_{l=1}^{L}\zeta\left(A_l\mid b\right).$$

By Kosorok (2008, Lemma 9.8), Giné and Nickl (2015, Theorem 3.6.9) and Nolan and Pollard (1987, Corollary 17), $\{\zeta(\cdot\mid b):b\in\mathbb{R}\}$ is Vapnik- Červonenkis(VC) type class of functions (see, e.g., Giné and Nickl, 2015, Definition 3.6.10 for its definition) with respect to a constant envelope. Therefore,

$$\frac{1}{\sqrt{L}}\sum_{l=1}^{L}\zeta\left(A_l\mid\cdot\right)\rightsquigarrow\sqrt{r_n}\cdot\mathbb{B}\left(G\left(\cdot\mid n\right)\right). \tag{43}$$

(41) follows from this result, (42), and the fact $\widehat{r}_n-r_n=O_p\left(L^{-1/2}\right)$. The second claim follows from (41) and (43). ∎

Let

$$\widetilde{\varphi}(\theta\mid n,\tau)=\int_0^{\gamma(\tau,\widehat{p}_n|\theta)}\left(\tau+\rho(u,\widehat{p}_n\mid\theta,n)\right)\mathrm{d}Q(u\mid n),$$

and we decompose:

$$\widehat{\varphi}(\theta\mid n,\tau)-\varphi(\theta\mid n,\tau)=\left\{\widehat{\varphi}(\theta\mid n,\tau)-\widetilde{\varphi}(\theta\mid n,\tau)\right\}+\left\{\widetilde{\varphi}(\theta\mid n,\tau)-\varphi(\theta\mid n,\tau)\right\}. \tag{44}$$

Denote

$$\pi\left(u\mid\theta,\tau,n\right):=\int_u^{\gamma(\tau,p_n|\theta)}\rho_1(t,p_n\mid\theta,n)\mathrm{d}Q(t\mid n)-\frac{\gamma(\tau,p_n\mid\theta)+p_n/(1-p_n)}{(n-1)\,\gamma_1(\tau,p_n\mid\theta)}\cdot q\left(\gamma(\tau,p_n\mid\theta)\mid n\right).$$

The following lemma establishes the uniform linearization for the first term on the right-hand side of (44).

Lemma 3. *Under the assumptions in the statement of Proposition 5, the following result holds uniformly in $(\theta,\tau)\in\Theta\times(0,1)$:*

$$\begin{aligned}\widehat{\varphi}(\theta\mid n,\tau)-\widetilde{\varphi}(\theta\mid n,\tau) &= \frac{1}{r_n}\cdot\frac{1}{L}\sum_{l:N_l=n}\sum_{i=1}^{N_l^*}\Bigg\{\mathbb{1}\left(G\left(B_{il}\mid n\right)\le\gamma(\tau,p_n\mid\theta)\right)\pi(G\left(B_{il}\mid n\right)\mid\theta,\tau,n)\\ &\quad-\int_0^{\gamma(\tau,p_n|\theta)}\pi(u\mid\theta,\tau,n)\mathrm{d}u\Bigg\}+o_p\left(L^{-1/2}\right).\end{aligned}$$

Proof of Lemma 3. For the first term on the right-hand side of (44)

$$\widehat{\varphi}(\theta\mid n,\tau)-\widetilde{\varphi}(\theta\mid n,\tau)=\int_0^{\gamma(\tau,\widehat{p}_n|\theta)}\left(\tau+\rho(u,\widehat{p}_n\mid\theta,n)\right)\mathrm{d}\left(\widehat{Q}(u\mid n)-Q(u\mid n)\right).$$

By integration by parts,

$$\begin{aligned}
\widehat{\varphi}(\theta \mid n, \tau)-\widetilde{\varphi}(\theta \mid n, \tau) &= (\tau+\rho(u, \widehat{p}_n \mid \theta, n))\left(\widehat{Q}(u \mid n)-Q(u \mid n)\right)\Big|_0^{\gamma(\tau, \widehat{p}_n \mid \theta)} \\
&\quad -\int_0^{\gamma(\tau, \widehat{p}_n \mid \theta)}\left(\widehat{Q}(u \mid n)-Q(u \mid n)\right) \rho_1(u, \widehat{p}_n \mid \theta, n) \mathrm{d} u \\
&= (\tau+\rho(\gamma(\tau, \widehat{p}_n \mid \theta), \widehat{p}_n \mid \theta, n))\left(\widehat{Q}(\gamma(\tau, \widehat{p}_n \mid \theta) \mid n)-Q(\gamma(\tau, \widehat{p}_n \mid \theta) \mid n)\right) \\
&\quad -(\tau+\rho(0, \widehat{p}_n \mid \theta, n))\left(\widehat{Q}(0 \mid n)-Q(0 \mid n)\right) \\
&\quad -\int_0^{\gamma(\tau, \widehat{p}_n \mid \theta)}\left(\widehat{Q}(u \mid n)-Q(u \mid n)\right) \rho_1(u, \widehat{p}_n \mid \theta, n) \mathrm{d} u \\
&=: T_1(\theta, \tau)-T_2(\theta, \tau)-T_3(\theta, \tau) .
\end{aligned}$$

Note that since $\widehat{Q}(0 \mid n)-Q(0 \mid n)=O_p(L^{-1})$, we have $T_2(\theta, \tau)=O_p(L^{-1})$, uniformly in $(\theta, \tau) \in \Theta \times(0,1)$. Therefore,

$$\widehat{\varphi}(\theta \mid n, \tau)-\widetilde{\varphi}(\theta \mid n, \tau)=T_1(\theta, \tau)-T_3(\theta, \tau)+O_p(L^{-1}), \tag{45}$$

uniformly in $(\theta, \tau) \in \Theta \times(0,1)$. By Lemma 2, Van der Vaart (2000, Lemma 21.4(ii)), and the functional delta method,

$$\widehat{Q}(u \mid n)-Q(u \mid n)=-\frac{\widehat{G}(Q(u \mid n) \mid n)-G(Q(u \mid n) \mid n)}{g(Q(u \mid n) \mid n)}+o_p\left(L^{-1/2}\right), \tag{46}$$

uniformly in $u \in(0,1)$. Combining this linearization with Lemma 1, we obtain

$$\begin{aligned}
T_1(\theta, \tau) &= -(\tau+\rho(\gamma(\tau, p_n \mid \theta), p_n \mid \theta, n)) q(\gamma(\tau, p_n \mid \theta) \mid n) \\
&\quad \times\left\{\widehat{G}(Q(\gamma(\tau, p_n \mid \theta) \mid n) \mid n)-G(Q(\gamma(\tau, p_n \mid \theta) \mid n) \mid n)\right\}+o_p\left(L^{-1/2}\right) \\
&= -(\tau+\rho(\gamma(\tau, p_n \mid \theta), p_n \mid \theta, n)) q(\gamma(\tau, p_n \mid \theta) \mid n) \\
&\quad \times \frac{1}{r_n} \cdot \frac{1}{L} \sum_{l: N_l=n} \sum_{i=1}^{N_l^*}\{\mathbb{1}(B_{il} \leq Q(\gamma(\tau, p_n \mid \theta) \mid n))-\gamma(\tau, p_n \mid \theta)\}+o_p\left(L^{-1/2}\right),
\end{aligned} \tag{47}$$

uniformly in $(\theta, \tau) \in \Theta \times(0,1)$.

By the triangle inequality,

$$\begin{aligned}
&\left|T_3(\theta, \tau)-\int_0^{\gamma(\tau, p_n \mid \theta)}\left(\widehat{Q}(u \mid n)-Q(u \mid n)\right) \rho_1(u, p_n \mid \theta, n) \mathrm{d} u\right| \\
\leq &\left|\int_0^{\gamma(\tau, \widehat{p}_n \mid \theta)}\left(\widehat{Q}(u \mid n)-Q(u \mid n)\right) \rho_1(u, p_n \mid \theta, n) \mathrm{d} u\right. \\
&\left.-\int_0^{\gamma(\tau, p_n \mid \theta)}\left(\widehat{Q}(u \mid n)-Q(u \mid n)\right) \rho_1(u, p_n \mid \theta, n) \mathrm{d} u\right|
\end{aligned}$$

$$+\left|\int_0^{\gamma(\tau,\widehat{p}_n|\theta)}\left(\widehat{Q}(u\mid n)-Q(u\mid n)\right)\left(\rho_1(u,\widehat{p}_n\mid\theta,n)-\rho_1(u,p_n\mid\theta,n)\right)\mathrm{d}u\right|, \tag{48}$$

The first term on the right-hand side of (48) is bounded by

$$\left\{\sup_{u\in(0,1)}\left|\left(\widehat{Q}(u\mid n)-Q(u\mid n)\right)\rho_1(u,p_n\mid\theta,n)\right|\right\}\left|\gamma(\tau,\widehat{p}_n\mid\theta)-\gamma(\tau,p_n\mid\theta)\right|=O_p\left(L^{-1}\right),$$

uniformly in $(\theta,\tau)\in\Theta\times(0,1)$. Using (46) together with Lemma 1 shows that the second term is $O_p\left(L^{-1}\right)$, uniformly in $(\theta,\tau)\in\Theta\times(0,1)$. Therefore,

$$T_3(\theta,\tau)=\int_0^{\gamma(\tau,p_n|\theta)}\left(\widehat{Q}(u\mid n)-Q(u\mid n)\right)\rho_1(u,p_n\mid\theta,n)\mathrm{d}u+o_p\left(L^{-1/2}\right),$$

uniformly in $(\theta,\tau)\in\Theta\times(0,1)$. Combining this expansion with (46),

$$\begin{aligned}T_3(\theta,\tau) &= -\frac{1}{r_n}\int_0^{\gamma(\tau,p_n|\theta)}\left\{\frac{1}{L}\sum_{l:N_l=n}\sum_{i=1}^{N_l^*}\left(\mathbb{1}\left(B_{il}\le Q(u\mid n)\right)-u\right)\right\}\rho_1(u,p_n\mid\theta,n)\mathrm{d}Q(u\mid n)\\ &\quad+o_p\left(L^{-1/2}\right).\end{aligned} \tag{49}$$

Combining the approximations in (47) and (49), we have

$$\begin{aligned}&T_1(\theta,\tau)-T_3(\theta,\tau)\\ =&\ \frac{1}{r_n}\cdot\frac{1}{L}\sum_{l:N_l=n}\sum_{i=1}^{N_l^*}\Bigg\{\int_0^{\gamma(\tau,p_n|\theta)}\left(\mathbb{1}\left(B_{il}\le Q(u\mid n)\right)-u\right)\rho_1(u,p_n\mid\theta,n)\mathrm{d}Q(u\mid n)\\ &-\left(\tau+\rho\left(\gamma(\tau,p_n\mid\theta),p_n\mid\theta,n\right)\right)q\left(\gamma(\tau,p_n\mid\theta)\mid n\right)\left(\mathbb{1}\left(B_{il}\le Q\left(\gamma(\tau,p_n\mid\theta)\mid n\right)\right)-\gamma(\tau,p_n\mid\theta)\right)\Bigg\}\\ &+o_p\left(L^{-1/2}\right),\end{aligned} \tag{50}$$

uniformly in $(\theta,\tau)\in\Theta\times(0,1)$. A direct calculation shows that

$$\tau+\rho\left(\gamma(\tau,p\mid\theta),p\mid\theta,n\right)=\frac{\gamma(\tau,p\mid\theta)+p/(1-p)}{(n-1)\,\gamma_1(\tau,p\mid\theta)}. \tag{51}$$

We can write

$$\begin{aligned}&\int_0^{\gamma(\tau,p_n|\theta)}\mathbb{1}\left(B_{il}\le Q(u\mid n)\right)\rho_1(u,p_n\mid\theta,n)\mathrm{d}Q(u\mid n)\\ =&\ \mathbb{1}\left(B_{il}\le Q\left(\gamma(\tau,p_n\mid\theta)\mid n\right)\right)\int_{G(B_{il}|n)}^{\gamma(\tau,p_n|\theta)}\rho_1(u,p_n\mid\theta,n)\mathrm{d}Q(u\mid n).\end{aligned}$$

By these calculations and (50),

$$\begin{aligned} T_1(\theta,\tau) - T_3(\theta,\tau) &= \frac{1}{r_n}\cdot\frac{1}{L}\sum_{l:N_l=n}\sum_{i=1}^{N_l^*}\Bigg\{\mathbb{1}\left(G\left(B_{il}\mid n\right)\le\gamma(\tau,p_n\mid\theta)\right)\pi(G\left(B_{il}\mid n\right)\mid\theta,\tau,n) \\ &\quad - \int_0^{\gamma(\tau,p_n\mid\theta)}\pi(u\mid\theta,\tau,n)\mathrm{d}u\Bigg\} + o_p\left(L^{-1/2}\right), \end{aligned}$$

uniformly in $(\theta,\tau)\in\Theta\times(0,1)$. Combining this result with (45) yields the lemma. ■

Denote

$$\Phi(p\mid\theta,\tau,n) := \int_0^{\gamma(\tau,p\mid\theta)}\left(\tau+\rho(u,p\mid\theta,n)\right)\mathrm{d}Q(u\mid n). \tag{52}$$

By the Leibniz rule and (51), we have

$$\begin{aligned} \Phi_1(p\mid\theta,\tau,n) &:= \frac{\partial}{\partial p}\Phi(p\mid\theta,\tau,n) \\ &= \frac{\gamma(\tau,p\mid\theta)+p/(1-p)}{(n-1)\,\gamma_1(\tau,p\mid\theta)}q\left(\gamma(\tau,p\mid\theta)\mid n\right)\gamma_2(\tau,p\mid\theta) \\ &\quad + \int_0^{\gamma(\tau,p\mid\theta)}\rho_2(u,p\mid\theta,n)\mathrm{d}Q(u\mid n). \end{aligned}$$

The following lemma establishes the uniform linearization for the second term on the right-hand side of (44).

Lemma 4. *Under the assumptions in the statement of Proposition 5, the following result holds uniformly in $(\theta,\tau)\in\Theta\times(0,1)$:*

$$\widetilde{\varphi}(\theta\mid n,\tau)-\varphi(\theta\mid n,\tau) = \frac{\Phi_1(p_n\mid\theta,\tau,n)}{\pi_n}\cdot\frac{1}{L}\sum_{l:N_l=n}\left\{\frac{N_l^*}{n}-(1-p_n)\right\}+o_p\left(L^{-1/2}\right).$$

Proof of Lemma 4. By using the notation defined in (52), we can write $\widetilde{\varphi}(\theta\mid n,\tau)=\Phi(\widehat{p}_n\mid\theta,\tau,n)$ and $\varphi(\theta\mid n,\tau)=\Phi(p_n\mid\theta,\tau,n)$. By mean value expansion and Lemma 1,

$$\widetilde{\varphi}(\theta\mid n,\tau)-\varphi(\theta\mid n,\tau)=\Phi_1(p_n\mid\theta,\tau,n)\left(\widehat{p}_n-p_n\right)+o_p\left(L^{-1/2}\right),$$

uniformly in $(\theta,\tau)\in\Theta\times(0,1)$. Combining this result with Lemma 1 yields the claim. ■

Proof of Proposition 5. Denote

$$\begin{aligned} \lambda\left(A_l\mid\theta,\tau,n\right) &:= \mathbb{1}\left(N_l=n\right)\sum_{i=1}^{N_l^*}\Bigg\{\mathbb{1}\left(G\left(B_{il}\mid n\right)\le\gamma(\tau,p_n\mid\theta)\right)\pi(G\left(B_{il}\mid n\right)\mid\theta,\tau,n) \\ &\quad - \int_0^{\gamma(\tau,p_n\mid\theta)}\pi(u\mid\theta,\tau,n)\mathrm{d}u\Bigg\}, \end{aligned}$$

where A_l is defined in the proof of Lemma 2. Then, by Lemma 3, we can write

$$\widehat{\varphi}(\theta \mid n, \tau) - \widetilde{\varphi}(\theta \mid n, \tau) = \frac{1}{r_n} \cdot \frac{1}{L} \sum_{l=1}^{L} \lambda\left(A_l \mid \theta, \tau, n\right) + o_p\left(L^{-1/2}\right), \tag{53}$$

uniformly in $(\theta, \tau) \in \Theta \times (0,1)$. By Andrews (1994, Theorem 2), $\{\rho_1(\cdot, p_n \mid \theta, n) : \theta \in \Theta\}$ is VC-type. By Ghosal et al. (2000, Lemmas A.1 and A.2), $\{\pi(\cdot \mid \theta, \tau, n) : (\theta, \tau) \in \Theta \times (0,1)\}$ is also VC-type. Combining this with Ghosal et al. (2000, Lemma A.1) shows that $\{\lambda\left(\cdot \mid \theta, \tau, n\right) : (\theta, \tau) \in \Theta \times (0,1)\}$ is VC-type. Therefore, by (53), Kosorok, 2008, Theorem 9.30(i), Kosorok, 2008, Corollary 9.32(i) and (v), Kosorok, 2008, Lemma 7.23(i) that

$$\sqrt{L}\left(\widehat{\varphi}(\theta_0 \mid n, \cdot) - \widetilde{\varphi}(\theta_0 \mid n, \cdot)\right) \rightsquigarrow \mathbb{G}_0\left(\cdot \mid n\right),$$

in $\ell^\infty\left(0,1\right)$. Lemma 4 implies

$$\widetilde{\varphi}(\theta_0 \mid n, \tau) - \varphi(\theta_0 \mid n, \tau) = -\frac{\phi(\tau, n)}{\pi_n} \cdot \frac{1}{L} \sum_{l : N_l = n} \left\{\frac{N_l^*}{n} - (1 - p_n)\right\} + o_p\left(L^{-1/2}\right), \tag{54}$$

uniformly in $\tau \in (0,1)$. Combining this result, (40), and the continuous mapping theorem gives

$$\sqrt{L}\left(\widetilde{\varphi}(\theta_0 \mid n, \cdot) - \varphi(\theta_0 \mid n, \cdot)\right) \rightsquigarrow \mathbb{G}_1\left(\cdot \mid n\right),$$

in $\ell^\infty\left(0,1\right)$. The leading terms on the right-hand sides of (53) (for any fixed τ) and (54) are asymptotically independent. This proves the proposition. ■

Proof of Theorem 1. Lemmas 3, and 4 imply that

$$\sup_{\theta \in \Theta} \left|\widehat{H}(\theta) - H(\theta)\right| \to_p 0.$$

Under global identifiability of θ_0, compactness of Θ, continuity of $H(\cdot)$, and the uniform consistency result given above, by Newey and McFadden (1994, Theorem 2.1), we have $\widehat{\theta} \to_p \theta_0$. Define $\widehat{M}(\theta) := -\widehat{H}(\theta)/2$ and $M(\theta) := -H(\theta)/2$. Then, $\widehat{\theta}$ is an approximate maximizer of $\widehat{M}(\theta)$ over $\theta \in \Theta$ and θ_0 uniquely maximizes $M(\theta)$ over $\theta \in \Theta$. Define

$$\begin{aligned} \delta(\theta \mid n, n', \tau) \;&:=\; \big(\widehat{\varphi}(\theta \mid n, \tau) - \widehat{\varphi}(\theta \mid n', \tau)\big) - \big(\widehat{\varphi}(\theta_0 \mid n, \tau) - \widehat{\varphi}(\theta_0 \mid n', \tau)\big) \\ &\quad - \big(\varphi(\theta \mid n, \tau) - \varphi(\theta \mid n', \tau)\big). \end{aligned}$$

Note that $\delta(\theta_0 \mid n, n', \tau) = 0$. Let $\sum_{n,n'}$ be shorthand notation for $\sum_{n,n' \in \mathcal{N} : n \neq n'}$. Define a modification of $\widehat{M}(\theta)$ by

$$\widetilde{M}(\theta) \;:=\; \widehat{M}(\theta) + \frac{1}{2} \sum_{n,n'} \int_0^1 \delta^2(\theta \mid n, n', \tau) \mathrm{d}\tau$$

$$+\sum_{n,n'}\int_0^1 \delta(\theta\mid n,n',\tau)\big(\widehat{\varphi}(\theta_0\mid n,\tau)-\widehat{\varphi}(\theta_0\mid n',\tau)\big)\mathrm{d}\tau. \tag{55}$$

Since $\delta(\theta_0\mid n,n',\tau)=0$, we have $\widetilde{M}(\theta_0)=\widehat{M}(\theta_0)$. By algebraic rearrangement, we have

$$\begin{aligned}
&\widetilde{M}(\theta)-\widetilde{M}(\theta_0)-M(\theta)\\
&+\sum_{n\neq n'}\int_0^1\big(\widehat{\varphi}(\theta_0\mid n,\tau)-\widehat{\varphi}(\theta_0\mid n',\tau)\big)\big(\varphi(\theta\mid n,\tau)-\varphi(\theta\mid n',\tau)\big)\mathrm{d}\tau\\
=\;&-\sum_{n\neq n'}\int_0^1 \delta(\theta\mid n,n',\tau)\big(\varphi(\theta\mid n,\tau)-\varphi(\theta\mid n',\tau)\big)\mathrm{d}\tau.
\end{aligned}\tag{56}$$

By (53), the fact that $\{\lambda(\cdot\mid\theta,\tau,n):(\theta,\tau)\in\Theta\times(0,1)\}$ is VC-type, and Andrews (1994, Theorem 1),

$$\sup_{|\theta-\theta_0|\leq\varepsilon_L}\big|\delta(\theta\mid n,n',\tau)\big|=o_p(L^{-1/2}), \tag{57}$$

uniformly in $\tau\in(0,1)$, for any positive ε_L such that $\varepsilon_L\downarrow 0$.

By Proposition 5, and (21), we have

$$\begin{aligned}
&\sup_{|\theta-\theta_0|\leq\varepsilon_L}\frac{\left|\int_0^1\big(\widehat{\varphi}(\theta_0\mid n,\tau)-\widehat{\varphi}(\theta_0\mid n',\tau)\big)\left\{\big(\varphi(\theta\mid n,\tau)-\varphi(\theta\mid n',\tau)\big)-\mathit{\Gamma}(n,n',\tau)(\theta-\theta_0)\right\}\mathrm{d}\tau\right|}{|\theta-\theta_0|}\\
=\;&o_p(L^{-1/2}),
\end{aligned}\tag{58}$$

for any positive ε_L such that $\varepsilon_L\downarrow 0$. By (57), and (21),

$$\sup_{|\theta-\theta_0|\leq\varepsilon_L}\frac{\left|\int_0^1\delta(\theta\mid n,n',\tau)\big(\varphi(\theta\mid n,\tau)-\varphi(\theta\mid n',\tau)\big)\mathrm{d}\tau\right|}{|\theta-\theta_0|}=o_p(L^{-1/2}).$$

Using this bound,

$$\sup_{|\theta-\theta_0|\leq\varepsilon_L}\frac{\left|\widetilde{M}(\theta)-\widetilde{M}(\theta_0)-M(\theta)+\left\{\sum_{n\neq n'}\int_0^1\big(\widehat{\varphi}(\theta_0\mid n,\tau)-\widehat{\varphi}(\theta_0\mid n',\tau)\big)\mathit{\Gamma}(n,n',\tau)\,\mathrm{d}\tau\right\}(\theta-\theta_0)\right|}{|\theta-\theta_0|}=o_p(L^{-1/2}), \tag{59}$$

for any positive ε_L such that $\varepsilon_L\downarrow 0$.

From (57), the definition in (55) and Proposition 5 we obtain

$$\sup_{|\theta-\theta_0|\leq\varepsilon_L}\left|\widetilde{M}(\theta)-\widehat{M}(\theta)\right|=o_p(L^{-1}), \tag{60}$$

for any positive ε_L such that $\varepsilon_L\downarrow 0$. Since $\widehat{\theta}\to_p\theta_0$, there exists some positive ε'_L such that $\varepsilon'_L\downarrow 0$

and $\left|\widehat{\theta}-\theta_0\right| \leq \varepsilon'_L$ with probability approaching one (see Pollard, 2002, Lemma 22). Therefore, $\widetilde{M}(\widehat{\theta}) \geq \widehat{M}(\widehat{\theta}) - o_p(L^{-1})$. Using the fact that $\widehat{M}(\widehat{\theta}) \geq \sup_{\theta\in\Theta} \widehat{M}(\theta) - o_p\left(L^{-1}\right)$ and (60), we have

$$\widetilde{M}(\widehat{\theta}) \geq \sup_{|\theta-\theta_0|\leq\varepsilon_L} \widetilde{M}(\theta) - o_p\left(L^{-1}\right), \tag{61}$$

for any positive ε_L such that $\varepsilon_L \downarrow 0$.

By (21) and straightforward algebra,

$$\begin{aligned} & \sum_{n\neq n'} \int_0^1 \left(\widehat{\varphi}(\theta_0 \mid n,\tau) - \widehat{\varphi}(\theta_0 \mid n',\tau)\right) \Gamma\left(n,n',\tau\right) \mathrm{d}\tau \\ = \quad & 2\sum_{n\in\mathcal{N}} \int_0^1 \Gamma\left(n,\tau\right) \left(\widehat{\varphi}(\theta_0 \mid n,\tau) - \varphi(\theta_0 \mid n,\tau)\right) \mathrm{d}\tau. \end{aligned} \tag{62}$$

Let $\{\mathbb{G}\left(\cdot \mid n\right)\}_{n\in\mathcal{N}}$ be mutually independent tight Gaussian processes which are distributed as described in the statement of Proposition 5. By Proposition 5,

$$\left\{\sqrt{L}\left(\widehat{\varphi}(\theta_0 \mid n,\cdot) - \varphi(\theta_0 \mid n,\cdot)\right)\right\}_{n\in\mathcal{N}} \rightsquigarrow \{\mathbb{G}\left(\cdot \mid n\right)\}_{n\in\mathcal{N}},$$

jointly in the product space $\left(\ell^\infty(0,1)\right)^{|\mathcal{N}|}$. Applying the continuous mapping theorem to this convergence,

$$\sqrt{L}\cdot\sum_{n\in\mathcal{N}} \int_0^1 \Gamma\left(n,\tau\right) \left(\widehat{\varphi}(\theta_0 \mid n,\tau) - \varphi(\theta_0 \mid n,\tau)\right) \mathrm{d}\tau \to_d \sum_{n\in\mathcal{N}} \int_0^1 \Gamma\left(n,\tau\right) \mathbb{G}\left(\tau \mid n\right) \mathrm{d}\tau, \tag{63}$$

where the limiting random variable in (63) is distributed as $\mathrm{N}\left(0, \sum_{n\in\mathcal{N}} V\left(n\right)\right)$.

Note that $M''\left(\theta_0\right) = -\bar{\Gamma}$. Then by (59), (61), (62), (63), and arguments in the proof of Newey and McFadden (1994, Theorem 7.1), we have

$$\widehat{\theta} = \theta_0 + \frac{2}{\bar{\Gamma}} \sum_{n\in\mathcal{N}} \int_0^1 \Gamma\left(n,\tau\right) \left(\widehat{\varphi}(\theta_0 \mid n,\tau) - \varphi(\theta_0 \mid n,\tau)\right) \mathrm{d}\tau + o_p\left(L^{-1/2}\right). \tag{64}$$

Slutsky's lemma applied to this expansion completes the proof. ■

Proof of Theorem 2. By Theorem 1, Lemma 1 and Taylor expansion, we have

$$\begin{aligned} \widehat{\kappa}_n - \kappa_n \quad = \quad & \int_0^1 k\left(t, \widehat{p}_n \mid \widehat{\theta}, n\right) \mathrm{d}\left(\widehat{Q}\left(t \mid n\right) - Q\left(t \mid n\right)\right) \\ & + \omega_2\left(n\right)\left(\widehat{p}_n - p_n\right) + \omega_\theta\left(n\right)\left(\widehat{\theta} - \theta_0\right) + o_p\left(L^{-1/2}\right). \end{aligned} \tag{65}$$

By using similar arguments as in the proof of Lemma 3, for the first term on the right-hand side of

(65), we have

$$\begin{aligned}&\int_0^1 k\left(t,\widehat{p}_n \mid \widehat{\theta},n\right)\mathrm{d}\left(\widehat{Q}(t\mid n)-Q(t\mid n)\right)\\ =\ & \frac{1}{r_n}\cdot\frac{1}{L}\sum_{l:N_l=n}\sum_{i=1}^{N_l^*}\left\{\int_{G(B_{il}|n)}^1 k_1(t,p_n\mid\theta_0,n)\,\mathrm{d}Q(t\mid n)-\int_0^1 tk_1(t,p_n\mid\theta_0,n)\,\mathrm{d}Q(t\mid n)\right\}\\ &+o_p\left(L^{-1/2}\right).\end{aligned} \tag{66}$$

Combining this result, (65), Lemma 1, and (64), we have

$$\widehat{\kappa}_n-\kappa_n=T_1+T_2+T_3+o_p\left(L^{-1/2}\right), \tag{67}$$

where

$$\begin{aligned}T_1 &:= \frac{1}{r_n}\cdot\frac{1}{L}\sum_{l:N_l=n}\sum_{i=1}^{N_l^*}\left\{\mathit{\Pi}\left(G\left(B_{il}\mid n\right)\mid n\right)-\int_0^1\mathit{\Pi}\left(u\mid n\right)\mathrm{d}u\right\},\\ T_2 &:= \frac{\omega_2(n)+\omega_\theta(n)\int_0^1\mathit{\Gamma}(n,\tau)\,\phi(\tau,n)\mathrm{d}\tau}{\pi_n}\cdot\frac{1}{L}\sum_{l:N_l=n}\left\{\frac{N_l^*}{n}-(1-p_n)\right\},\\ T_3 &:= \omega_\theta(n)\frac{2}{\bar{\mathit{\Gamma}}}\sum_{n'\in\mathcal{N}:n'\neq n}\int_0^1\mathit{\Gamma}\left(n',\tau\right)\left(\widehat{\varphi}(\theta_0\mid n',\tau)-\varphi(\theta_0\mid n',\tau)\right)\mathrm{d}\tau.\end{aligned}$$

The leading terms of T_1, T_2, T_3 are asymptotically independent. By the central limit theorem, $T_1 \to_d \mathrm{N}(0,\Sigma_1)$ and $T_2 \to_d \mathrm{N}(0,\Sigma_2)$. From (63) we have $T_3 \to_d \mathrm{N}(0,\Sigma_3)$. Combining these limits with (67), and Slutsky's lemma gives the theorem. ■

Appendix C Proofs of results in Section 5

This appendix collects the auxiliary lemmas underlying the cube-root asymptotics for the marginal distribution and the proof of Theorem 3.

Lemma 5. *Under the assumptions in the statement of Theorem 3, for each $\tau\in(0,1)$, uniformly for t in compact sets,*

$$L^{2/3}\int_\tau^{\tau+tL^{-1/3}}\left\{\widehat{Q}(u\mid n)-Q(u\mid n)\right\}\mathrm{d}u=o_p(1).$$

Lemma 6. *Under the assumptions in the statement of Theorem 3, for each $\tau\in(0,1)$,*

$$\begin{aligned}&\mathbb{Q}(\cdot\mid n)\rightsquigarrow\frac{q(\tau\mid n)}{\sqrt{r_n}}\mathbb{W},\ \textit{in}\ \ell^\infty[-K,K],\ \textit{where}\\ &\mathbb{Q}(t\mid n):=L^{2/3}\left\{\widehat{Q}\left(\tau+tL^{-1/3}\mid n\right)-\widehat{Q}(\tau\mid n)-Q\left(\tau+tL^{-1/3}\mid n\right)+Q(\tau\mid n)\right\},\end{aligned}$$

for every $K > 0$.

Lemma 7. *Under the assumptions in the statement of Theorem 3, for each* $\tau \in (0,1)$,

$$\mathbb{I}(\cdot \mid n) \rightsquigarrow a_n(\tau)\,\mathbb{W}, \text{ in } \ell^\infty[-K,K], \text{ where}$$
$$\mathbb{I}(t \mid n) := L^{2/3}\left\{\widehat{I}^*\left(\tau + tL^{-1/3} \mid p_n\right) - \widehat{I}^*(\tau \mid p_n) - I^*\left(\tau + tL^{-1/3} \mid p_n\right) + I^*(\tau \mid p_n)\right\},$$

for every $K > 0$.

Lemma 8. *Suppose that Assumptions 1–6 and 7 are satisfied. For fixed* $n \in \mathcal{N}$ *and fixed* $\tau \in (0,1)$,

$$\sqrt[3]{L}\left(\widehat{Q}^*(\tau \mid p_n) - Q^*(\tau \mid p_n)\right) \to_d 2a_n(\tau)^{2/3}\, b_n(\tau)^{1/3}\, Z.$$

$\widehat{Q}^*(\cdot \mid p_n)$ *thus has an exactly rescaled Chernoff limit.*

The proofs of Lemmas 5–8 follow by arguments similar to those of Luo and Wan (2018) based on the GCM switch relation together with a localized empirical process argument and are therefore omitted. The only modifications relative to the no-entry case are the markup numerator $\tau + p_n/(1-p_n)$ and the the effective sample size r_n, which produces the factor $r_n^{-1/2}$ in Lemma 6. The $\sqrt{L}$-consistent first-stage estimators $\widehat{\theta}$ and $\widehat{p}_n$ are negligible asymptotically.

Proof of Theorem 3. By (12), $Q^*(\gamma(\tau, p_n \mid \theta_0) \mid p_n) = Q(\tau)$ for every $n \in \mathcal{N}$, so

$$\widehat{Q}(\tau) - Q(\tau) = \frac{1}{|\mathcal{N}|}\sum_{n\in\mathcal{N}}\left\{\widehat{Q}^*\left(\gamma\left(\tau, \widehat{p}_n \mid \widehat{\theta}\right) \mid p_n\right) - Q^*(\gamma(\tau, p_n \mid \theta_0) \mid p_n)\right\}.$$

Since $\gamma\left(\tau, \widehat{p}_n \mid \widehat{\theta}\right) - \gamma(\tau, p_n \mid \theta_0) = O_p\left(L^{-1/2}\right)$, which is $o_p\left(L^{-1/3}\right)$ in the local coordinate, the random evaluation rank may be replaced by $\gamma(\tau, p_n \mid \theta_0)$: the localized process converges to a limit with almost surely continuous paths and is evaluated at the converging argument, while the smooth shift $Q^*\left(\gamma\left(\tau, \widehat{p}_n \mid \widehat{\theta}\right) \mid p_n\right) - Q^*(\gamma(\tau, p_n \mid \theta_0) \mid p_n) = O_p\left(L^{-1/2}\right)$ is itself negligible. Each term then obeys Lemma 8 and converges to $c_n(\tau)\, Z_n$. The leading local scores in Lemma 7 are sums over the mutually exclusive events $\{N_l = n\}$, so their cross-covariances vanish and the driving Brownian motions are mutually independent. Therefore, the limiting Chernoff variables Z_n are independent across $n \in \mathcal{N}$. The first-stage estimators $\widehat{\theta}$ and $\widehat{p}_n$ are $\sqrt{L}$-consistent and therefore negligible asymptotically. ■